Stable-Π Partitions of Graphs

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Abstract

For a set of graphs Π, the STABLE-Π problem asks whether, given a graph G, we can find an independent set S in G, such that $G - S \in \Pi$. For instance, if Π is the set of all bipartite graphs, STABLE-Π coincides with VERTEX 3-COLOURABILITY, and if Π is the set of 1-regular graphs, the problem is known as EFFICIENT EDGE DOMINATION. Numerous other examples of the STABLE-Π problem have been studied in the literature. In the present contribution, we systematically study the STABLE-Π problem with respect to the speed (a term meaning size) of Π. In particular, we show that for all hereditary classes Π with a subfactorial speed of growth, STABLE-Π is solvable in polynomial time. We then explore the problem for minimal hereditary factorial classes Π. Contrary to the conjecture proposed in [18], the complexity of STABLE-Π turns out to be polynomial for nearly all minimal hereditary factorial classes Π. On the other hand, if we do not require Π to be hereditary, the complexity of the problem can jump to NP-completeness.

Key words: stable-Π partition, hereditary property, speed of graph property, factorial property, polynomial-time, NP-complete

1. Introduction

In this paper, all graphs are undirected, with no loops or parallel edges. A graph is *bipartite*, *co-bipartite* or *split* if its vertex set can be partitioned into two independent sets, two cliques or a clique and an independent set, respectively. If X is a set of vertices in a graph G, we use $G - X$ to denote the graph obtained from G by deleting every vertex in X. We use $G[X]$ to denote the subgraph of G induced by X, i.e. the graph $G - (V(G) \setminus X)$. If G and H are graphs, then G is H-free if it does not contain an induced subgraph isomorphic to H. We use $2K_2$ to denote the graph consisting of two disjoint edges, and P_4 to denote the chordless path on four vertices.

Let Π be a graph property (or graph class), i.e. a set of graphs closed under isomorphism. A property Π is *hereditary* if it is closed under taking induced subgraphs, and it is *additive* if it is closed under taking disjoint unions of graphs.

For a property Π, the STABLE-Π problem asks, given a graph G, to determine whether G has an independent set S such that $G - S \in \Pi$. The family of STABLE-Π problems has been extensively studied in the literature (see e.g. [5, 6, 7, 8, 11, 13, 14, 15, 20]) and includes many important representatives such as VERTEX 3-COLOURABILITY, in which case Π is the set of all bipartite graphs, and EFFICIENT EDGE DOMINATION (also known as DOMINATING INDUCED MATCHING), in which case Π is the set of all 1-regular graphs. Both of these examples represent algorithmically hard, i.e. NP-complete, problems. The STABLE-Π problem is also NP-complete for various other properties Π such as forests or trivially perfect graphs [5]. More generally, the problem remains NP-complete for any additive hereditary property Π other than the set of edgeless graphs [16].

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On the other hand, for some properties Π, the STABLE-Π problem can be solved in polynomial time. This is the case, for instance, if Π is the class of co-bipartite graphs [5] or the class of complete bipartite graphs [4]. The case of co-bipartite graphs was generalised independently in [2] and [9] to arbitrary hereditary properties Π which are of bounded independence number and which can be recognised in polynomial time. The case where Π is the class of complete bipartite graphs has also received a wide generalisation. To describe this generalisation, let us observe that the class of complete bipartite graphs is quite small. In the terminology of [3], it is *subfactorial*, i.e. for any constant $c > 0$, Π has less than n^{cn} labelled graphs on n vertices, if n is sufficiently large. Subfactorial graph properties have a simple structural characterisation (see Theorem 1). This was used in [18] to prove that the STABLE-Π problem is polynomial-time solvable for any subfactorial hereditary property Π of bipartite graphs.

In the present paper, we further generalise this result to arbitrary subfactorial hereditary properties Π (not necessarily of bipartite graphs). We then switch to hereditary properties with a *factorial* speed of growth, i.e. those containing at least $n^{c_1 n}$ and at most $n^{c_2 n}$ labelled graphs on n vertices for some constants $c_1, c_2 > 0$, when n is sufficiently large. The family of factorial graph properties is much wider and contains many classes of theoretical or practical importance. For instance the classes of threshold graphs, line graphs, permutation graphs, and interval graphs are factorial and all classes of graphs of bounded vertex degree, of bounded clique-width and all proper minor closed graph classes have at most factorial speed of growth.

The family of factorial hereditary classes is very rich and varied, but there are only a few such classes for which the complexity of the STABLE-Π problem is known. It is therefore natural to focus on the simplest classes in this family, namely those that are *minimal* (when ordered by set inclusion). There are exactly nine such classes [1, 3]. Three of them are subclasses of bipartite graphs:

- $\mathcal{M}_1$ **bipartite matching graphs:** *graphs partitionable into two independent sets, where the edges between them form a matching (equivalently, graphs of maximum degree one)*
- $\mathcal{M}_2$ **bipartite almost complete graphs:** *graphs partitionable into two independent sets such that each vertex has at most one non-neighbour in the opposite part*
- $\mathcal{M}_3$ **chain graphs:** *bipartite $2K_2$-free graphs*

Three other minimal factorial classes are subclasses of co-bipartite graphs: these are precisely the classes of complements of graphs in $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, which we denote by $\overline{\mathcal{M}_1}$, $\overline{\mathcal{M}_2}$, and $\overline{\mathcal{M}_3}$, respectively. The remaining three minimal factorial classes are subclasses of split graphs. They are also closely related to $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ and can be obtained from graphs in these classes by converting one of the independent sets in the bipartition into a clique. We denote these classes as follows:

- $\mathcal{M}_4$ **split matching graphs:** *graphs partitionable into a clique and an independent set, where the edges between them form a matching*
- $\overline{\mathcal{M}_4}$ **complements of split matching graphs:** *graphs partitionable into a clique and an independent set so that each vertex has at most one non-neighbour in the opposite part*
- $\mathcal{M}_5$ **threshold graphs:** *split P_4-free graphs*

It is known that STABLE-$\mathcal{M}_1$ is an NP-complete problem [17, 19], while STABLE-$\mathcal{M}_5$ is solvable in polynomial time [5]. For the remaining seven minimal factorial classes, the complexity of the problem was unknown and we study it in the present paper.

The borderline between factorial and subfactorial properties was also studied in [21] for the following problem associated with a hereditary class Π of bipartite graphs: given a bipartite graph G, find the largest induced subgraph of G that belongs to Π. Yannakakis [21] showed that this problem is solvable in polynomial time if Π is a subfactorial hereditary class, and is NP-hard otherwise (except for the case when Π coincides with the class of all bipartite graphs, in which case the problem is trivial). Inspired by this result, Lozin conjectured [18] that the STABLE-Π problem is NP-complete for all hereditary factorial classes of bipartite graphs, including the three minimal hereditary factorial classes. Contrary to this conjecture, we

Π	Stable-Π		Stable-Π^S	
$\mathcal{M}_1$	NP-C	[17, 19]	NP-C	[12]
$\overline{\mathcal{M}_1}$	P	Thm 6	P	Thm 15
$\mathcal{M}_2$	P	Thm 14	NP-C	Thm 19
$\overline{\mathcal{M}_2}$	P	Thm 6	P	Thm 15
$\mathcal{M}_3$	open		n/a	
$\overline{\mathcal{M}_3}$	P	Thm 6	n/a	
$\mathcal{M}_4$	P	Thm 7	NP-C	Thm 17
$\overline{\mathcal{M}_4}$	P	Thm 8	NP-C	Thm 18
$\mathcal{M}_5 = \overline{\mathcal{M}_5}$	P	[5]	n/a	

Table 1: Summary of complexity results.

show that Stable-Π is solvable in polynomial time for nearly all minimal hereditary factorial classes Π (not necessarily bipartite).

Let us emphasise that these nine minimal classes of graphs are hereditary and most of the instances of the Stable-Π problem that have been studied in the literature deal with hereditary properties Π. On the other hand, some important examples of the problem appear in the context of non-hereditary properties Π. We already mentioned Efficient Edge Domination, which is equivalent to Stable-Π when Π is the set of 1-regular graphs. We denote the class of 1-regular graphs by $\mathcal{M}_1^S$. Observe that this set is a restriction of the class $\mathcal{M}_1$. More precisely, $\mathcal{M}_1$ is the hereditary closure of the set of 1-regular graphs (i.e. it is the set containing all 1-regular graphs and all their induced subgraphs). In the same spirit, we define $\mathcal{M}_2^S$ to be the class of graphs partitionable into two independent sets such that each vertex has exactly one non-neighbour in the opposite part and define $\mathcal{M}_4^S$ to be the class of graphs partitionable into a clique and an independent set such that every vertex in one part has exactly one neighbour in the opposite part. As before, we write $\overline{\mathcal{M}_1^S}$, $\overline{\mathcal{M}_2^S}$ and $\overline{\mathcal{M}_4^S}$ to denote the classes of graphs whose complements are in $\mathcal{M}_1^S$, $\mathcal{M}_2^S$ and $\mathcal{M}_4^S$, respectively.

We find that for some minimal factorial classes Π for which Stable-Π can be solved in polynomial time, the restriction to Π^S leads to an NP-complete problem. A summary of our results is given in Table 1.

2. Preliminaries

A *graph property*, or *graph class* is any set Π of simple graphs closed under isomorphism. The *graph-complement* $\overline{\Pi}$ of a property Π is defined as $\overline{\Pi} = \{\overline{G} \mid G \in \Pi\}$. A graph property is *hereditary* if it is closed under vertex removal, or equivalently, under taking induced subgraphs. A hereditary graph property Π is *factorial* if there exist positive constants c_1, c_2, N such that $n^{c_1 n} \le |\Pi_n| \le n^{c_2 n}$ when $n > N$, where Π_n denotes the set of n-vertex labelled graphs in Π. A class is *subfactorial* if for every $c > 0$, $|\Pi_n| \le n^{cn}$ when n is sufficiently large. The structure of subfactorial classes is rather simple and can be characterised as follows.

Let M be a symmetric $\{0,1\}$-matrix with t rows and t columns, for some $t > 0$. Let b be a function $b : \{1, \dots, t\} \to \mathbb{N} \cup \{\infty\}$. We write $\mathcal{P}(M, b)$ to denote the set of all graphs G that admit a vertex partition into t sets $V_1, \dots, V_t$ such that

(I) each set V_i contains at most $b(i)$ elements,

(II) V_i is a clique of G if $M(i,i) = 1$; otherwise V_i is an independent set of G, and

(III) for distinct sets V_i, V_j, either every vertex of V_i is adjacent to every vertex of V_j if $M(i,j) = 1$; or there are no edges between V_i and V_j if $M(i,j) = 0$.

Theorem 1. [1, 3] *For every subfactorial hereditary class Π, there is a finite collection (depending only on Π) of matrices $M_1, \dots, M_k$ and functions $b_1, \dots, b_k$ such that Π is precisely the union of $\mathcal{P}(M_i, b_i)$ where $i = 1, \dots, k$.*

Note that the original theorem in [3] states that the above holds for sufficiently large graphs. We can omit this condition by including the adjacency matrices of all smaller graphs into the collection $M_1, \dots, M_k$.

3. Subfactorial properties

Theorem 2. *For any subfactorial hereditary property* Π*, the* STABLE-Π *problem is solvable in polynomial time.*

PROOF. By Theorem 1, it suffices to explain how to solve the STABLE-Π problem in polynomial time when $\Pi = \mathcal{P}(M, b)$ where M is a symmetric $\{0,1\}$-matrix M with t rows and t columns, and b is a function $b : \{1, \ldots, t\} \to \mathbb{N} \cup \{\infty\}$.

Without loss of generality, we may assume that M contains no repeated rows or columns, since by removing any such row (and the corresponding column), and adjusting b, we can obtain an equivalent description of Π, with smaller t. Indeed, if, for instance, last two rows of M are the same, then we can instead consider the matrix M' obtained from M by removing its last row and column, and define b' as follows: $b'(i) = b(i)$ for all $i < t-1$, and $b'(t-1) = b(t-1) + b(t)$ if neither $b(t-1)$ nor $b(t)$ is ∞; otherwise $b'(t-1) = \infty$. Then clearly $\mathcal{P}(M', b') = \mathcal{P}(M, b) = \Pi$.

With this assumption, we can now explain how to solve the STABLE-Π problem. Namely, given a graph $G = (V, E)$, we want to determine if there is a partition $V = S \cup R$, such that S is an independent set of G and $G[R] \in \Pi$. We call any partition $V_1, \ldots, V_t$ of R *canonical* if it satisfies conditions (I)-(III), as defined before the statement of Theorem 1, and call the subsets in a canonical partition *bags*.

Without loss of generality, we shall try to find a canonical partition of R where all bags are non-empty. Indeed, if we cannot find such a partition, we may in turn try to solve the problem for all submatrices of M in place of M, and there are only constantly many such submatrices.

We start by setting the set R and all bags $V_1, \ldots, V_t$ to be empty. Then, we proceed as follows. For each i such that $b(i) \neq \infty$, we choose a non-empty subset of at most $b(i)$ vertices of G to be the bag V_i, and include V_i into R. We also designate some $v_i \in V_i$ to be the *representative* of V_i. Similarly, for each i where $b(i) = \infty$, we pick one vertex v_i of G to be the representative of the bag V_i, and move v_i to V_i and R. Note that there are $O(n^c)$ ways to accomplish all of this, where $c = t + \sum_{b(i) \neq \infty} b(i)$ is a constant.

We consider every such set of choices in turn. Recall that each such choice produces initial sets $V_1, \ldots, V_t$ forming a partition of R and representatives $v_1, \ldots, v_t$ where $v_i \in V_i$ for $i \in \{1, \ldots, t\}$. If the partition of R into $V_1, \ldots, V_t$ is not canonical, we disregard this choice. Otherwise, we use 2SAT to find a set S disjoint from R (if exists) such that $G - S \in \Pi$. Namely, we use 2SAT to assign each vertex in $V \setminus R$ either to S or to one of the bags V_i with $b(i) = \infty$. (We consider the bags V_i with $b(i) \neq \infty$ to be fixed by the initial choice, and thus we do not allow assigning further vertices to these bags.) We proceed as follows.

For each vertex $v \in V \setminus R$, we define a *candidate* bag V_i to be a bag such that including v into V_i and R results in a canonical partition of R. We observe that there can be at most one candidate bag for v, since we assume that M contains no identical rows. To see this, note that it is possible to include v in V_i if for every j (possibly $j = i$), we have $vv_j \in E(G)$ if $M(i,j) = 1$, and have $vv_j \notin E(G)$ if $M(i,j) = 0$. Clearly, there cannot be distinct bags V_i, $V_{i'}$ satisfying this for v, since this would imply that the rows i and i' of M were identical, which we assumed not to be the case.

The instance of 2SAT is constructed as follows. For every vertex $v \in V \setminus R$, we create a Boolean variable x_v. For each $v \in V \setminus R$, we add the unit clause (x_v) if either there is no candidate bag for v, or if the candidate bag V_i for v is such that $b(i) \neq \infty$. For distinct vertices $u, v \in V \setminus R$, we add the clause $(\overline{x}_u \vee \overline{x}_v)$ if u, v are adjacent. Similarly, for distinct $u, v \in V \setminus R$, we add the clause $(x_u \vee x_v)$ if u, v are adjacent (resp. non-adjacent), but either they have the same candidate bag which is an independent set (resp. a clique), or their candidate bags are distinct but non-adjacent (resp. adjacent).

The resulting instance of 2SAT has polynomial size and thus can be solved in polynomial time. In particular, if the instance is satisfiable, then $S = \{v \mid x_v = \mathit{true}\}$ is an independent set and $G - S \in \Pi$. If the instance is not satisfiable, then no solution exists for the fixed sets V_i and representatives v_i that we chose at the start. In this case we try another set of choices. If no such initial choice of sets succeeds, then we declare that there is no independent set in G whose removal produces a graph in Π.

This concludes the proof. $\square$

4. Minimal factorial properties

In this section, we discuss the complexity of STABLE-Π for minimal factorial hereditary classes Π. We investigate each case as set out in the introduction. Note that all these problems are clearly in the class NP as testing any of the properties $\mathcal{M}_1, \ldots, \mathcal{M}_5$ can be easily done in polynomial (in fact, linear) time. The same holds for the restricted cases discussed later, in Section 5.

The following cases have already been established in the literature.

Theorem 3. [17, 19] *The* STABLE-$\mathcal{M}_1$ *problem is NP-complete.*

Theorem 4. [5] *The* STABLE-$\mathcal{M}_5$ *problem is solvable in polynomial time.*

Further results in this section are based on the notion of Sparse-Dense partitions.

Theorem 5. (Sparse-Dense Theorem) [2, 9] *For all positive integers k, l, there exists a polynomial time algorithm that, given a graph G, constructs all partitions of its vertex set into sets A, B such that $G[A]$ contains no independent set of size k and $G[B]$ contains no clique of size l.*

Namely, there are at most $n^{2R(k,l)-2}$ such partitions of an n-vertex graph G and all can be enumerated in time $O\big(n^{2R(k,l)+\max\{k,l\}}\big)$, where $R(k,l)$ denotes the Ramsey number of k and l.

Theorem 6. *The* STABLE-$\overline{\mathcal{M}_1}$, STABLE-$\overline{\mathcal{M}_2}$, *and* STABLE-$\overline{\mathcal{M}_3}$ *problems are solvable in polynomial time.*

PROOF. Let $\Pi \in \{\overline{\mathcal{M}_1}, \overline{\mathcal{M}_2}, \overline{\mathcal{M}_3}\}$. All three problems ask to partition the vertices of the input graph G into one independent set V_1, and a co-bipartite graph V_1' (consisting of two cliques V_2 and V_3). By Theorem 5, there are only polynomially many such partitions of $V(G)$ and all of them can be found in polynomial time. For each such partition, we test whether the co-bipartite subgraph of G induced by V_1' is in Π. This yields a polynomial-time algorithm. □

The following two theorems are proved in a similar way to how Theorem 4 was proved in [5].

Theorem 7. *The* STABLE-$\mathcal{M}_4$ *problem is solvable in polynomial time.*

PROOF. We rephrase the problem as: given a graph G, decide whether the vertices of G can be partitioned into three sets V_1, V_2, V_3 such that V_3 is a clique, V_1 and V_2 are independent sets and every vertex in V_2 has at most one neighbour in V_3 and vice-versa.

Let G be the input graph. By Theorem 5, we can find, in polynomial time, the collection $\mathcal{P}$ of all partitions of the vertex set of G into a clique C and a set X such that $G[X]$ contains no clique of size three. Note that if G admits a STABLE-$\mathcal{M}_4$ partition V_1, V_2, V_3, then the partition $C = V_3$, $X = V_1 \cup V_2$ is a partition in $\mathcal{P}$. Thus to solve the problem, we try all partitions C, X in $\mathcal{P}$ by setting $V_3 = C$ and testing whether X can be split into V_1, V_2 so that V_1, V_2, V_3 is a STABLE-$\mathcal{M}_4$ partition of G.

Let C, X be a partition from $\mathcal{P}$. We construct the following instance $\mathcal{I}$ of 2SAT.

(i) Create a variable x_v for every vertex $v \in X$,

(ii) for every edge $uv \in E(G[X])$, add the clauses $(x_u \vee x_v)$ and $(\overline{x_u} \vee \overline{x_v})$,

(iii) for every pair of vertices $u, v \in X$ with a common neighbour in C, add the clause $(\overline{x_u} \vee \overline{x_v})$, and

(iv) for every vertex $v \in X$ such that v has at least two neighbours in C, add the clauses $(\overline{x_v} \vee \overline{a})$ and $(\overline{x_v} \vee a)$, where a is a new variable.

We claim that $\mathcal{I}$ has a satisfying assignment if and only if G admits a STABLE-$\mathcal{M}_4$ partition V_1, V_2, V_3 such that $V_3 = C$ and $V_1 \cup V_2 = X$.

Suppose that the instance $\mathcal{I}$ has a satisfying truth assignment φ. Namely, φ is a mapping from the variables of $\mathcal{I}$ to $\{true, false\}$ such that in every clause C_j, there is at least one literal that φ evaluates to *true* (where the value $\varphi(\overline{z})$ is defined as the negation of $\varphi(z)$, for any variable z).

Define $V_1 = \{v \mid \varphi(x_v) = false\}$ and $V_2 = \{v \mid \varphi(x_v) = true\}$. We claim that V_1, V_2, V_3 is a STABLE-$\mathcal{M}_4$ partition of G. Indeed, by (ii), V_1 and V_2 are independent sets; by (iii), no two vertices in V_2 have a common neighbour in V_3; and by (iv), every vertex from V_2 has at most one neighbour in V_3.

Conversely, let V_1, V_2, V_3 be a STABLE-$\mathcal{M}_4$ partition of G where $V_3 = C$. We define a truth assignment for $\mathcal{I}$ as follows. We set $\varphi(x_v) = false$ if $v \in V_1$ and $\varphi(x_v) = true$ if $v \in V_2$. For each of the new variables a defined in (iv) above, we set $\varphi(a) = true$. We claim that φ is a satisfying truth assignment for $\mathcal{I}$. Indeed, all clauses defined in (ii) are satisfied, since V_1 and V_2 are independent sets. Also, all clauses defined in (iii) are satisfied since every vertex in V_3 has at most one neighbour in V_2. Similarly, every vertex in V_2 has at most one neighbour in V_3 implying that all clauses in (iv) are satisfied. Thus $\mathcal{I}$ is satisfied by φ.

This concludes the proof. □

A similar argument works for the complementary class and results in the following theorem.

Theorem 8. *The* STABLE-$\overline{\mathcal{M}_4}$ *problem is solvable in polynomial time.*

PROOF. Similarly to the proof of Theorem 7, we can rephrase the problem as: given a graph G, decide whether the vertices of G can be partitioned into three sets V_1, V_2, V_3 such that V_3 is a clique, V_1 and V_2 are independent sets and every vertex in V_2 has at most one non-neighbour in V_3 and vice-versa.

Again, defining $\mathcal{P}$ as before, we solve the problem by trying all partitions C, X in $\mathcal{P}$. For each such partition we set $V_3 = C$ and test whether X can be split into V_1, V_2 so that V_1, V_2, V_3 is a STABLE-$\overline{\mathcal{M}_4}$ partition of G.

Let G'_C be the graph obtained from G by complementing (i.e. replacing edges by non-edges and vice versa) the edges between C and X. Now G has a STABLE-$\overline{\mathcal{M}_4}$ partition with $V_3 = C$ if and only if G'_C has a STABLE-$\mathcal{M}_4$ partition with $V_3 = C$. Indeed, if $V_1 \cup V_2$ is a partition of X, then $G[C]$ is a clique and $G[V_1]$, $G[V_2]$ are independent sets if and only if $G'_C[C]$ is a clique and $G'_C[V_1]$, $G'_C[V_2]$ are independent sets. Further, each vertex in V_2 (resp. V_3) has at most one non-neighbour in V_3 (resp. V_2) in G if and only if it has at most one neighbour in V_3 (resp. V_2) in G'_C.

We now reduce the problem to an equivalent instance of 2SAT as in the proof of Theorem 7. This concludes the proof. □

We are left with the case of the STABLE-$\mathcal{M}_2$ problem, which needs more work. We solve it in the following section.

4.1. *The* STABLE-$\mathcal{M}_2$ *problem*

In this section, we prove that the STABLE-$\mathcal{M}_2$ problem is solvable in polynomial time. We cast the problem for the complement and solve (in polynomial time) a more general version with lists as follows.

An *instance* of the problem is a pair (G, ℓ) where G is a graph and $\ell : V(G) \to 2^{\{1,2,3\}}$. We say that $\ell(v)$ is the *list* belonging to the vertex v. For $S \subseteq \{1,2,3\}$, we let U^ℓ_S denote the set of vertices in G with $\ell(v) = S$.

Given an instance (G, ℓ), we seek to partition $V(G)$ into three cliques V_1, V_2, V_3 such that

- each vertex in V_2 has at most one neighbour in V_3,
- each vertex in V_3 has at most one neighbour in V_2, and
- for all $\alpha \in \{1,2,3\}$, each $v \in V_\alpha$ satisfies $\alpha \in \ell(v)$.

If such a partition exists, we call it a *solution* for (G, ℓ). Note that if the list of some vertex is empty, then there is no solution for the problem instance. Thus for the rest of the proof, we assume that $U^\ell_\emptyset = \emptyset$.

To solve the problem, we consider several special cases and reduce the general case to these cases in polynomial time. An important role in this process will be played by a particular reduction algorithm (Algorithm 1) that applies a series of necessary conditions simplifying the problem, eventually leading to a handful of special cases. We shall apply this reduction algorithm throughout all subsequent steps as necessary. For instance, note that the test in Line 2 makes sure that each of the sets $U^\ell_{\{1\}}$, $U^\ell_{\{2\}}$, $U^\ell_{\{3\}}$ is a clique, since otherwise there is clearly no solution for the input instance. Another test, in Line 5, is derived from the fact that a vertex in V_2 can only have one neighbour in V_3 and vice versa; thus a vertex in $U^\ell_{\{1,2\}}$

Algorithm 1: Reduction algorithm

Input: Instance (G,ℓ) where G is a graph and $\ell(v) : V(G) \to 2^{\{1,2,3\}}$
Output: A reduced instance (G,ℓ)

1 **for** $\alpha \in \{1,2,3\}$ **do**
2   **if** *for* $u \in U^{\ell}_{\{\alpha\}}$, *there exists* $v \in V(G) \setminus N(u)$ *with* $\alpha \in \ell(v)$ **then**
    remove α from $\ell(v)$ and **goto** 1
3 **for** $(\alpha,\beta) \in \Big\{(2,3),(3,2)\Big\}$ **do**
4   **if** *for* $u \in U^{\ell}_{\{\alpha\}}$, *there exists* $v \in N(u) \cap U^{\ell}_{\{\beta\}}$ **then**
    for all $w \in N(u) \setminus \{v\}$ with $\beta \in \ell(w)$, **remove** β from $\ell(w)$
    for all $w \in N(v) \setminus \{u\}$ with $\alpha \in \ell(w)$, **remove** α from $\ell(w)$
    remove u,v from G and **goto** 1
5   **if** *there exists* $v \in U^{\ell}_{\{1,\beta\}}$ *with* $|N(v) \cap U^{\ell}_{\{\alpha\}}| \geq 2$ **then**
    remove β from $\ell(v)$ and **goto** 1
6   **if** *for* $u \in U^{\ell}_{\{\alpha\}}$, *there are* $v,w \in N(u) \cap U^{\ell}_{\{1,\beta\}}$ *where* $(N(v) \setminus N(w)) \cap U^{\ell}_{\{1,\alpha\}} \neq \emptyset$ **then**
    remove β from $\ell(v)$ and **goto** 1
7   **if** *for* $u \in U^{\ell}_{\{\alpha\}}$, *there are* $v,w \in N(u) \cap U^{\ell}_{\{1,\beta\}}$ *and* $x \in U^{\ell}_{\{1,\alpha\}}$ *with* $v,w \notin N(x)$ **then**
    remove 1 from $\ell(x)$ and **goto** 1
8   **if** *for* $u \in V(G)$ *with* $1 \in \ell(u)$, *the set* $U^{\ell}_{\{1,\alpha\}} \setminus N(u)$ *is not a clique* **then**
    remove 1 from $\ell(u)$ and **goto** 1
9   **if** *for* $u \in V(G)$ *with* $\beta \in \ell(u)$, *the subgraph* $G\Big[N(u) \cap U^{\ell}_{\{1,\alpha\}}\Big]$ *contains an induced 4-cycle* **then**
    remove β from $\ell(u)$ and **goto** 1
10 **return** (G,ℓ)

can be safely moved to $U^{\ell}_{\{1\}}$ if it has more than one neighbour in $U^{\ell}_{\{3\}}$. Other reduction rules follow from a similar analysis of possible cases (a full explanation can be found in the proof below).

We say that an instance (G,ℓ) is *reduced*, if it is the result of Algorithm 1. We have the following claim.

Lemma 9. *Let (G,ℓ) be an instance and let (G',ℓ') be the result of applying Algorithm 1 to (G,ℓ). Then there exists a solution for (G,ℓ) if and only if there exists a solution for (G',ℓ').*

PROOF. Note that if $x \in U^{\ell}_{\{i\}}$ for some $i \in \{1,2,3\}$, then in any solution (V_1,V_2,V_3) of the instance, we have $x \in V_i$. Using this we justify the reductions rules as follows.

Line 2: Let $\alpha \in \{1,2,3\}$. Since V_α must be a clique in any solution, if $u \in U^{\ell}_{\{\alpha\}}$ and u,v are not adjacent, then $v \notin V_\alpha$ for any solution for (G,ℓ).

In the remainder of the proof, we have $\alpha = 2$ and $\beta = 3$, or $\alpha = 3$ and $\beta = 2$.

Line 4: If u,v are adjacent for some $u \in U^{\ell}_{\{\alpha\}}$ and $v \in U^{\ell}_{\{\beta\}}$, then in any valid solution, these must be two matched vertices of V_2 and V_3. In this case v must be the unique neighbour of u in V_β and u must be the unique neighbour of v in V_α. We can therefore remove either α or β from the list of each vertex in $N(u) \cup N(v) \setminus \{u,v\}$, as appropriate. We then remove u and v from G. The resulting instance has a solution if and only if the original one does.

Line 5: In any solution, if $v \in V_\beta$, then v can have at most one neighbour in V_α.

Line 6: Suppose $u \in U^{\ell}_{\{\alpha\}}$, such that $v,w \in N(u) \cap U^{\ell}_{\{1,\beta\}}$ and $z \in (N(v) \setminus N(w)) \cap U^{\ell}_{\{1,\alpha\}}$. If there were a solution in which $v \in V_\beta$, then since $u \in V_\alpha$ and every vertex in V_α can have at most one neighbour in V_β

and vice versa, we must have $w, z \in V_1$. But this is impossible, since w, z are not adjacent. This contradiction implies that v cannot be in V_β.

Line 7: Suppose $u \in U^\ell_{\{\alpha\}}, x \in U^\ell_{\{1,\alpha\}}$ and $v, w \in (N(u) \setminus N(x)) \cap U^\ell_{\{1,\beta\}}$. Then in any solution we must have $u \in V_\alpha$. Since u can have at most one neighbour in V_β, at least one of v, w must be in V_1. But V_1 is a clique and v, w are nonadjacent to x. Thus $x \notin V_1$.

Line 8: Suppose $u \in V(G)$ with $1 \in \ell(u)$ and $v, w \in U^\ell_{\{1,\alpha\}} \setminus N(u)$ with v, w non-adjacent. Since for any solution, V_i must be a clique for $i \in \{1, 2, 3\}$, exactly one of v, w must be in V_1 and the other in V_α. But u is non-adjacent to both v and w, so $u \notin V_1$.

Line 9: Suppose $\beta \in \ell(u)$. In any solution, if $u \in V_\beta$ then $N(u) \cap V_1$ must be a clique and u can have at most one neighbour in V_α. The 4-cycle is neither a clique, nor is it partitionable into a clique and a single vertex. Thus if a 4-cycle is an induced subgraph of $G[N(u) \cap U^\ell_{\{1,\alpha\}}]$, then any solution must satisfy $u \notin V_\beta$.

This completes all the reduction rules and the claim follows. □

Note that Algorithm 1 has polynomial running time. This allows us to assume that the instance we consider is always reduced. (If not, we use Algorithm 1 to produce an equivalent reduced instance.)

Assuming this, we consider some special cases of the problem, which we will later use as steps in finding a solution for the general problem.

Lemma 10. *If there exists a solution (V_1, V_2, V_3) for the reduced instance (G, ℓ), such that there is no edge between a vertex in V_2 and a vertex in V_3, it can be found in polynomial time.*

PROOF. This amounts to finding a partition of $\overline{G}$ into an independent set and a complete bipartite graph, in a way that respects the lists of the vertices. This can been solved in polynomial time [9]. □

Lemma 11. *If $U^\ell_{\{1,2,3\}} = U^\ell_{\{2,3\}} = \emptyset$, and $U^\ell_{\{1,2\}} = \emptyset$ or $U^\ell_{\{1,3\}} = \emptyset$, and the instance is reduced, the problem can be solved in polynomial time.*

PROOF. We may assume by symmetry that $U^\ell_{\{1,3\}} = \emptyset$ and we reduce the problem to an instance of 2SAT constructed as follows.

- For each vertex $x \in U^\ell_{\{1,2\}}$, introduce a new variable v_x.
- For all $z \in U^\ell_{\{3\}}$ and all $x, y \in N(z) \cap U^\ell_{\{1,2\}}$, add the clause $(\neg v_x \vee \neg v_y)$.
- For all $x, y \in U^\ell_{\{1,2\}}$ with $xy \notin E(G)$, add the clauses $(v_x \vee v_y)$, $(\neg v_x \vee \neg v_y)$.

Since (G, ℓ) is a reduced instance, it has a solution if and only if the above instance of 2SAT is satisfiable. In particular, if φ is a satisfying assignment, the following sets (V_1, V_2, V_3) form a solution for (G, ℓ).

$$V_1 = U^\ell_{\{1\}} \cup \{x \mid \varphi(v_x) = false\} \qquad V_2 = U^\ell_{\{2\}} \cup (U^\ell_{\{1,2\}} \setminus V_1) \qquad V_3 = U^\ell_{\{3\}}$$ □

Lemma 12. *If $U^\ell_{\{1,2,3\}} = U^\ell_{\{2,3\}} = \emptyset$ and $U^\ell_{\{1,2\}}$, $U^\ell_{\{1,3\}}$ are cliques of G, and the instance is reduced, the problem can be solved in polynomial time.*

PROOF. We show that the following is a solution for (G, ℓ).

$$V_1 = U^\ell_{\{1\}} \cup U^\ell_{\{1,2\}} \cup \bigcup_{\substack{u \in U^\ell_{\{2\}} \\ |N(u) \cap U^\ell_{\{1,3\}}| \geq 2}} \left(N(u) \cap U^\ell_{\{1,3\}}\right) \qquad V_2 = U^\ell_{\{2\}} \qquad V_3 = U^\ell_{\{3\}} \cup (U^\ell_{\{1,3\}} \setminus V_1)$$

Indeed, note that the instance (G, ℓ) is reduced. By Line 2 of Algorithm 1 and the fact that $U^\ell_{\{1,3\}}$ is a clique, we conclude that V_2 and V_3 must be cliques. By Line 4 of Algorithm 1 and the definition of V_1 and V_3, every vertex in V_2 has at most one neighbour in V_3. By Lines 4 and 5 of Algorithm 1, each vertex of V_3 has at most one neighbour in V_2. By Line 2 of Algorithm 1 and since $U^\ell_{\{1,2\}}, U^\ell_{\{1,3\}}$ are cliques, we need only verify that every vertex in $V_1 \cap U^\ell_{\{1,2\}}$ is adjacent to every vertex in $V_1 \cap U^\ell_{\{1,3\}}$. We therefore assume that these sets are not empty. Let $u \in U^{\ell}_{\{2\}}$ and $v, w \in N(u) \cap U^{\ell}_{\{1,3\}}$. By Line 7 of Algorithm 1, any vertex in $U^{\ell}_{\{1,2\}}$ must be adjacent to at least one of v or w. But by Line 6 of Algorithm 1, the vertices v, w have the same neighbourhood in $U^{\ell}_{\{1,2\}}$. Thus every vertex of $U^{\ell}_{\{1,2\}}$ must be adjacent to every vertex of $V_1 \cap U^{\ell}_{\{1,3\}}$. We therefore conclude that V_1 is indeed a clique. □

We can now generalise Lemmas 11 and 12 as follows.

Lemma 13. *If $U^{\ell}_{\{1,2,3\}} = U^{\ell}_{\{2,3\}} = \emptyset$, and the problem instance is reduced, the problem can be solved in polynomial time.*

PROOF. Assume that $U^{\ell}_{\{1,2,3\}} = U^{\ell}_{\{2,3\}} = \emptyset$, but Lemma 11 does not apply. Thus $U^{\ell}_{\{1,2\}} \neq \emptyset$ and $U^{\ell}_{\{1,3\}} \neq \emptyset$.

We fix any $u \in U^{\ell}_{\{1,2\}}$. Then we either do nothing, or choose $w \in N(u) \cap U^{\ell}_{\{1,3\}}$ and set $\ell(w) = \{3\}$. After that, we remove 3 from $\ell(v)$ for each $v \in N(u)$ that belongs to a non-trivial (≥ 2 vertices) connected component of $\overline{G}\left[U^{\ell}_{\{1,3\}}\right]$ unless that component contains w (if w exists). We then apply Algorithm 1 to ensure that we have a reduced instance.

If after these modifications $U^{\ell}_{\{1,3\}}$ is still non-empty, we similarly fix $u' \in U^{\ell}_{\{1,3\}}$, do nothing or set $\ell(w') = \{2\}$ for some $w' \in N(u') \cap U^{\ell}_{\{1,2\}}$, and then remove 2 from $\ell(v)$ for each $v \in N(u') \cap U^{\ell}_{\{1,2\}}$ in a non-trivial component of $\overline{G}\left[U^{\ell}_{\{1,2\}}\right]$ unless that component contains w' (if w' exists). Afterwards, we again apply Algorithm 1 to ensure that we have a reduced instance.

We try all possible choices for w and w', creating $O(n^2)$ instances. We show that the initial instance has a solution if and only if (at least) one of these $O(n^2)$ instances has. After that, we will show that each of these $O(n^2)$ instances can be solved in polynomial time, which will conclude the proof.

Claim 1. *(G, ℓ) has a solution if and only if (at least) one of the $O(n^2)$ instances has a solution.*

Clearly, if one of the $O(n^2)$ instances has a solution, then this is also a solution for (G, ℓ), since during the construction of the instances, we only remove elements from lists.

Conversely, let V_1, V_2, V_3 be a solution for (G, ℓ). Let $H = G[U^{\ell}_{\{1,3\}}]$, i.e. H denotes the subgraph of G induced by $U^{\ell}_{\{1,3\}}$, and consider the vertex $u \in U^{\ell}_{\{1,2\}}$ that we fix.

Case (i): Suppose that $u \in V_1$. There are two possibilities to consider. First, suppose that there exists a neighbour of u that is in V_3 and also in some non-trivial connected component of $\overline{H}$. Consider the instance where we choose w to be this neighbour. (We shall henceforth refer to it as the "modified" instance.) In this instance, we remove 3 from each neighbour of u in $V(H) = U^{\ell}_{\{1,3\}}$ that belongs to a non-trivial connected component of $\overline{H}$ unless that component contains w.

We claim that each such neighbour v belongs to V_1. Suppose otherwise. Then v belongs to V_3, since $\ell(v) = \{1, 3\}$. Recall that v is in a non-trivial connected component of $\overline{H}$. Thus it has a neighbour z in $\overline{H}$. We conclude that z is non-adjacent to v in H, and hence, in G. If z is also non-adjacent to u, then z can be neither in V_1 nor in V_3, as these are both cliques. But then V_1, V_2, V_3 cannot be a solution for (G, ℓ) as $\ell(z) = \{1, 3\}$. So, we conclude that z is adjacent to u.

Now, recall that w is also in a non-trivial connected component of $\overline{H}$. So, w has a neighbour x in this component, and we conclude that $xw \notin E(G)$. This implies $ux \in E(G)$ as otherwise V_1, V_2, V_3 is not a solution. But now x, z, w, v induce a 4-cycle in the neighbourhood of u, which is impossible by Line 9 of Algorithm 1. (For this, recall that (G, ℓ) is a reduced instance and that the connected component of $\overline{H}$ containing w and x is different from the one containing v and z.)

This proves that V_1, V_2, V_3 is also a solution to the modified instance. As this is one of the $O(n^2)$ instances, we are done.

So, we may assume that each neighbour of u in $V_3 \cap V(H)$ is itself a connected component (isolated vertex) of $\overline{H}$. In this case, we consider the instance where we do not choose w (referred to as the "modified" instance). In this instance, we remove 3 from each neighbour of u in $V(H)$ that belongs to a non-trivial connected component of $\overline{H}$. By our assumption, this does not modify the lists of those neighbours of u that are in $V_3 \cap V(H)$. Thus V_1, V_2, V_3 is a solution to the modified instance, and we are done.

Case (ii): Suppose that $u \in V_2$. If u has a neighbour in $V_3 \cap V(H)$, consider the instance where w is chosen to be this neighbour (referred to as the "modified" instance). In this instance, we remove 3 from each neighbour of u in a non-trivial connected component of $\overline{H}$ unless that component contains w. Clearly, any such vertex v cannot belong to V_3, since then u would have two neighbours in V_3, which is impossible. Thus V_1, V_2, V_3 is also a solution to the modified instance, and we are done.

Finally, suppose that u has no neighbour in $V_3 \cap V(H)$, and consider the instance where we do not choose w. Again, we remove 3 from every neighbour of u in a non-trivial component of $\overline{H}$, and conclude that V_1, V_2, V_3 is a solution to this modified instance, since we assume that $N(u) \cap V(H) \cap V_3 = \emptyset$. This completes all cases.

This proves that one of the choices for w must succeed if (G, ℓ) has a solution. By a symmetric argument, it follows that, for an appropriate choice of w, one of the choices for w' (if at all we consider w') must also succeed. This concludes the proof of Claim 1.

Now, we explain how to solve each of the $O(n^2)$ instances in polynomial time. Consider one of the $O(n^2)$ instances (G^+, ℓ^+). We constructed this instance from the initial instance (G, ℓ), by fixing a vertex u and choosing w (or not), and then fixing a vertex u' (if possible) and choosing w' (or not). We also reduced this instance using Algorithm 1. We now prove that the sets $U^{\ell^+}_{\{1,2\}}$ and $U^{\ell^+}_{\{1,3\}}$ are either both cliques of G^+, or one them is empty. In other words, we show that Lemma 11 or 12 can be applied to the instance (G^+, ℓ^+).

Claim 2. *$U^{\ell^+}_{\{1,2\}}$ and $U^{\ell^+}_{\{1,3\}}$ are both cliques of G^+, or one of $U^{\ell^+}_{\{1,2\}}$, $U^{\ell^+}_{\{1,3\}}$ is empty.*

If one of $U^{\ell^+}_{\{1,2\}}$, $U^{\ell^+}_{\{1,3\}}$ is empty, we are done. So we may assume that both $U^{\ell^+}_{\{1,2\}}$ and $U^{\ell^+}_{\{1,3\}}$ are non-empty. For contradiction, assume first that $U^{\ell^+}_{\{1,3\}}$ contains non-adjacent vertices v, v'. As ℓ^+ is a restriction of ℓ and since $U^{\ell}_{\{1,2,3\}} = \emptyset$, we conclude that v, v' are also vertices in $U^{\ell}_{\{1,3\}}$. Again, let H denote $G[U^{\ell}_{\{1,3\}}]$.

First, we observe that u is adjacent to at least one of v, v'. Indeed, if u is non-adjacent to both v and v', then 1 was removed from $\ell(u)$ in Line 8 of Algorithm 1 (recall that (G, ℓ) is a reduced instance). This is impossible as $\ell(u) = \{1, 2\}$. By symmetry, we shall assume that u is adjacent to v.

Now, if w was not chosen when constructing (G^+, ℓ^+), then 3 was removed from all neighbours of u in non-trivial connected components of $\overline{H}$. One of these components contains both v and v' as they are non-adjacent, and so 3 was removed from $\ell(v)$ when constructing ℓ^+ (recall that we assume that u is adjacent to v). However, this is impossible, since $\ell^+(v) = \{1, 3\}$. We similarly arrive at a contradiction when w is chosen, but it is not a vertex of the connected component of $\overline{H}$ containing v. So we conclude that w was chosen from the connected component of $\overline{H}$ containing v. But now, we have that either $v = w$, or, since (G^+, ℓ^+) is reduced, 1 or 3 was removed from $\ell(v)$ in Line 2 at some point when running Algorithm 1 to produce the instance (G^+, ℓ^+). This is, of course, impossible as $\ell(w) = \{3\}$ and $\ell^+(v) = \{1, 3\}$. This concludes the argument for $U^{\ell^+}_{\{1,3\}}$.

The argument for $U^{\ell^+}_{\{1,2\}}$ is similar, using u' and w'. For this, note that u' exists, since we assume that $U^{\ell^+}_{\{1,3\}}$ is non-empty. This concludes the proof of Claim 2.

In conclusion, we deduce that each of the $O(n^2)$ instances can be decided in polynomial time (by Lemma 11 or 12). Thus, by Claim 1, the initial instance can be solved in polynomial time as required. □

We are finally ready to discuss the general case and prove the main theorem of this section.

Theorem 14. *The* STABLE-$\mathcal{M}_2$ *problem is solvable in polynomial time.*

PROOF. First, we test whether or not we are in the situation of Lemma 10. If so, we find a solution for (G, ℓ) using [9]. If not, we conclude that if there is a solution (V_1, V_2, V_3) for (G, ℓ), then there must exist

$u \in V_2$ and $v \in V_3$ with $uv \in E(G)$. We try all possible choices for such a pair u, v. This reduces the problem to solving $O(n^2)$ separate instances. For each such choice u, v, we set $\ell(u) = \{2\}$, $\ell(v) = \{3\}$, and run Algorithm 1. If the list of some vertex becomes empty, we reject this choice of u, v. Otherwise, we observe that the resulting reduced instance (G', ℓ') satisfies $U^{\ell'}_{\{1,2,3\}} = U^{\ell'}_{\{2,3\}} = \emptyset$. So we can apply Lemma 13 to (G', ℓ'), which determines in polynomial time if there is a solution for (G, ℓ). This concludes the proof. □

We close this section by emphasising that the above algorithm actually solves the more general *list* version of the problem where each vertex carries a list $\ell(v)$ such that v can only be assigned to V_α where $\alpha \in \ell(v)$. An analogous extension to solving the list version is easily possible for all the other polynomial cases we described in Theorems 4, 6, and 7.

5. Restricted Minimal Factorial Properties

First, we briefly examine the polynomial-time cases. Using essentially the same arguments as in the proof of Theorem 6, we obtain the following theorem.

Theorem 15. *The* STABLE-$\overline{\mathcal{M}^S_1}$ *and* STABLE-$\overline{\mathcal{M}^S_2}$ *problems are solvable in polynomial time.*

All the remaining cases are hard. We discuss them in separate claims. All the subsequent proofs will be essentially along the same lines and based on the following useful lemma.

The problem ONE-IN-THREE-3SAT asks to find an assignment of truth values to variables of a 3-CNF formula (conjunction of 3-literal disjunctions = *clauses*) such that in every clause exactly one of the three literals is true. The problem is well-known to be NP-complete [10].

Lemma 16. *Any non-empty instance of* ONE-IN-THREE-3SAT *can be transformed in polynomial time to an equivalent instance of* ONE-IN-THREE-3SAT *such that*

(i) There is no clause of the form $(X \vee X \vee Y)$ or $(X \vee \overline{X} \vee Y)$ where X, Y are (not necessarily distinct) literals.
(ii) If X appears in some clause, then $\overline{X}$ also appears in some clause.
(iii) Every literal appears at least twice in the instance.
(iv) There are at least 4 clauses and at least 4 variables in the instance.

PROOF. Apply the following steps in order. First, for each clause of the form $(X \vee X \vee Y)$, replace it by the clauses $(u \vee v \vee X)$, $(\overline{u} \vee \overline{v} \vee X)$, $(w \vee z \vee \overline{Y})$, $(\overline{w} \vee \overline{z} \vee \overline{Y})$, where u, v, w, z are new variables. Next, for each clause of the form $(X \vee \overline{X} \vee Y)$, replace it by the clauses $(u \vee v \vee Y)$, $(\overline{u} \vee \overline{v} \vee Y)$, where u, v are new variables. Then, for each literal X, add the clauses $(u \vee v \vee X)$, $(\overline{u} \vee w \vee \overline{X})$, $(\overline{v} \vee w \vee z)$, $(v \vee \overline{w} \vee z)$, $(v \vee w \vee \overline{z})$, where u, v, w, z are new variables. Note that since the original instance was non-empty, the new instance must now have at least 4 clauses and at least 4 variables. Finally, make a copy of each clause, i.e. make each clause appear twice in the instance.

It is easy to see that the instance produced in this way is equivalent to the original instance and satisfies all the conditions of the lemma. □

Theorem 17. *The* STABLE-$\mathcal{M}^S_4$ *problem is NP-complete.*

PROOF. We can rephrase the problem as follows: given a graph G, decide whether the vertices of G can be partitioned into 3 sets V_1, V_2, V_3 such that V_3 is a clique, V_1 and V_2 are independent sets and the edges between V_2 and V_3 form a perfect matching.

The proof proceeds by reduction from ONE-IN-THREE-3SAT. Consider an instance $\mathcal{I}$ of the problem, namely the instance consists of m clauses $C_1, \ldots, C_m$ containing variables $v_1, \ldots, v_n$. We may assume it satisfies the properties listed in Lemma 16. Let J_i denote the set of indices j such that v_i appears in C_j. Let $\overline{J_i}$ denote the indices j such that $\overline{v_i}$ appears in C_j.

For the instance $\mathcal{I}$, we construct the graph $G_{\mathcal{I}}$ as follows. First, we create a complete graph on vertices $y_1, \ldots, y_m$. Then for every occurrence of a variable v_i (resp. $\overline{v_i}$) in a clause C_j, we add a new vertex $x_{i,j}$

(resp. $\overline{x_{i,j}}$) and we add an edge between y_j and $x_{i,j}$ (resp. $\overline{x_{i,j}}$). Finally, we add an edge between $x_{i,j}$ and $\overline{x_{i,\ell}}$ for all $i \in \{1, \ldots, n\}$, all $j \in J_i$ and all $\ell \in \overline{J_i}$.

We prove that $G_{\mathcal{I}}$ admits a STABLE-$\mathcal{M}_4^S$ partition if and only if $\mathcal{I}$ has a satisfying truth assignment (as an instance of ONE-IN-THREE-3SAT).

Suppose that the instance $\mathcal{I}$ has a satisfying truth assignment φ. In other words, φ is a mapping from $\{v_1, \ldots, v_n\}$ to $\{true, false\}$ such that for every clause C_j, φ evaluates exactly one of the literals in C_j to *true*, where $\varphi(\overline{v_i})$ is defined as the negation of $\varphi(v_i)$. Let us define a partition of $V(G_{\mathcal{I}})$ as follows:

$$
\begin{aligned}
V_1 &= \left\{x_{i,j} \;\middle|\; j \in J_i \wedge \varphi(v_i) = false\right\} \cup \left\{\overline{x_{i,j}} \;\middle|\; j \in \overline{J_i} \wedge \varphi(v_i) = true\right\},\\
V_2 &= \left\{x_{i,j} \;\middle|\; j \in J_i \wedge \varphi(v_i) = true\right\} \cup \left\{\overline{x_{i,j}} \;\middle|\; j \in \overline{J_i} \wedge \varphi(v_i) = false\right\},\\
V_3 &= \left\{y_j \;\middle|\; j \in \{1, \ldots, m\}\right\}.
\end{aligned}
$$

It is not difficult to verify that V_1 and V_2 are independent sets of $G_{\mathcal{I}}$, that V_3 is a clique, and that the edges between V_2 and V_3 form a perfect matching. Indeed, each vertex y_j in V_3 is adjacent to a unique vertex $x_{i,j}$ or $\overline{x_{i,j}}$ in V_2, namely the one for which v_i, resp. $\overline{v_i}$ is the literal of C_j that φ evaluates to *true*. Thus $G_{\mathcal{I}}$ admits a STABLE-$\mathcal{M}_4^S$ partition as required.

Conversely, suppose that $G_{\mathcal{I}}$ admits a STABLE-$\mathcal{M}_4^S$ partition. In other words, there exists a partition of $V(G_{\mathcal{I}})$ into three sets V_1, V_2, V_3 such that V_1, V_2 are independent sets, V_3 is a clique, and the edges between V_2 and V_3 form a perfect matching.

First, we show that we must have $V_3 = \{y_j \mid j \in \{1, \ldots, m\}\}$. By Lemma 16, there are at least four y_j's. Thus, since V_1 and V_2 are independent sets, V_3 must contain at least two y_j's. This implies that V_3 contains no $x_{i,j}$ or $\overline{x_{i,j}}$, since each has at most one neighbour in $\{y_1, \ldots, y_m\}$ and V_3 is a clique. It also implies that if $y_j \in V_2$ for some j, then y_j has at least 2 neighbours in V_3, which is a contradiction. Finally, suppose that $y_j \in V_1$ for some j. Consider a neighbour $z \notin \{y_1, \ldots, y_m\}$ of y_j. (Note that z is $x_{i,j}$ or $\overline{x_{i,j}}$ for some i and there are exactly three such vertices). Then z is not in V_3, since V_3 contains no $x_{i,j}$ or $\overline{x_{i,j}}$. Also, z cannot be in V_1, since V_1 is independent. Thus z must be in V_2. But z has a unique neighbour in $\{y_1, \ldots, y_m\}$, namely y_j, and hence, z does not have a neighbour in V_3, a contradiction. This proves that $V_3 = \{y_1, \ldots, y_m\}$.

Now, we define the following truth assignment $\varphi : \{v_1, \ldots, v_n\} \to \{true, false\}$. For each $i \in \{1, \ldots, n\}$, we set $\varphi(v_i) = true$ if $x_{i,j} \in V_2$ for some j, and set $\varphi(v_i) = false$ otherwise. We prove that φ is a satisfying truth assignment for the instance $\mathcal{I}$, which will conclude the proof.

Using the assignment φ, we prove that

$$
\begin{aligned}
V_1 &= \left\{x_{i,j} \;\middle|\; j \in J_i \wedge \varphi(v_i) = false\right\} \cup \left\{\overline{x_{i,j}} \;\middle|\; j \in \overline{J_i} \wedge \varphi(v_i) = true\right\},\\
V_2 &= \left\{x_{i,j} \;\middle|\; j \in J_i \wedge \varphi(v_i) = true\right\} \cup \left\{\overline{x_{i,j}} \;\middle|\; j \in \overline{J_i} \wedge \varphi(v_i) = false\right\}.
\end{aligned}
$$

To show this, recall that for each $i \in \{1, \ldots, n\}$, every $x_{i,j}$ is adjacent to every $\overline{x_{i,\ell}}$ where $j \in J_i$ and $\ell \in \overline{J_i}$. Thus if $\varphi(v_i) = true$, then $x_{i,j} \in V_2$ for some j which implies $\overline{x_{i,\ell}} \in V_1$ for all $\ell \in \overline{J_i}$, since V_2 is an independent set. Therefore, $x_{i,j} \in V_2$ for all $j \in J_i$, since V_1 is an independent set. Similarly, if $\varphi(v_i) = false$, then $x_{i,j} \in V_1$ for all $j \in J_i$, and hence, $\overline{x_{i,\ell}} \in V_2$ for all $\ell \in \overline{J_i}$.

Now, consider a clause C_j. Recall that $y_j \in V_3$, and hence, it has exactly one neighbour $x_{i,j}$ or $\overline{x_{i,j}}$ in V_2 corresponding to the literal v_i, resp. $\overline{v_i}$ in C_j, which φ evaluates to *true* by the above. So, all other neighbours $x_{i',j}$ or $\overline{x_{i',j}}$ of y_j belong to V_1 and thus correspond to literals $v_{i'}$, resp. $\overline{v_{i'}}$ which φ evaluates to *false*. This proves that C_j is satisfied by φ, and thus, proves that φ is a satisfying truth assignment.

This concludes the proof. □

Similar constructions also work for the following two cases.

Theorem 18. *The* STABLE-$\overline{\mathcal{M}_4^S}$ *problem is NP-complete.*

PROOF. Again, we rephrase the problem as: given a graph G, decide whether the vertices of G can be partitioned into 3 sets V_1, V_2, V_3 such that V_3 is a clique, V_1 and V_2 are independent sets and the edges between V_2 and V_3 form the complement of a perfect matching.

The proof will now follow essentially the same steps as the proof of Theorem 17. We proceed by reduction from ONE-IN-THREE-3SAT.

Consider an instance $\mathcal{I}$ of the problem, namely the instance consists of m clauses $C_1, \dots, C_m$ containing variables $v_1, \dots, v_n$. Again, we may assume it satisfies the properties listed in Lemma 16. We define J_i to be the set of indices j such that v_i appears in C_j, and define $\overline{J_i}$ to be the set of indices j such that $\overline{v_i}$ appears in C_j.

For the instance $\mathcal{I}$, consider the graph $G_{\mathcal{I}}$ constructed in the proof of Theorem 17. Let $G_{\mathcal{I}}^+$ be the graph obtained from $G_{\mathcal{I}}$ by complementing the edges between $\{y_1, \dots, y_m\}$ and the rest of the graph. Namely, for each $i \in \{1, \dots, m\}$, the vertex y_i is adjacent to $z \notin \{y_1, \dots, y_m\}$ in $G_{\mathcal{I}}^+$ if and only if y_i is not adjacent to z in $G_{\mathcal{I}}$. All other edges remain the same.

We prove that $G_{\mathcal{I}}^+$ admits a STABLE-$\overline{\mathcal{M}_4^S}$ partition if and only if $\mathcal{I}$ has a satisfying truth assignment (as an instance of ONE-IN-THREE-3SAT).

For the forward direction, we note that, by the proof of Theorem 17, if $G_{\mathcal{I}}$ admits a STABLE-$\mathcal{M}_4^S$ partition V_1, V_2, V_3, then $V_3 = \{y_1, \dots, y_m\}$. Thus, this is also a STABLE-$\overline{\mathcal{M}_4^S}$ partition of $G_{\mathcal{I}}^+$. This proves that if $\mathcal{I}$ has a satisfying truth assignment, then $G_{\mathcal{I}}^+$ admits a STABLE-$\overline{\mathcal{M}_4^S}$ partition.

Conversely, suppose that $G_{\mathcal{I}}^+$ admits a STABLE-$\overline{\mathcal{M}_4^S}$ partition. Namely, let V_1, V_2, V_3 be a partition of $V(G_{\mathcal{I}})$ such that V_1, V_2 are independent sets, V_3 is a clique, and the edges between V_2 and V_3 form the complement of a perfect matching.

We shall prove that $V_3 = \{y_1, \dots, y_m\}$. By the construction of $G_{\mathcal{I}}^+$, this will imply that V_1, V_2, V_3 is also a STABLE-$\mathcal{M}_4^S$ partition of $G_{\mathcal{I}}$. Thus, by the proof of Theorem 17, this will allow us to conclude that $\mathcal{I}$ has a satisfying truth assignment.

Consider a vertex y_j. By Lemma 16, there is a variable v_i such that neither v_i nor $\overline{v_i}$ appears in the clause C_j. Moreover, v_i appears as a literal in at least two clauses, say C_{j_1} and C_{j_2}, and $\overline{v_i}$ appears in two other clauses, say C_{j_3} and C_{j_4}. This implies that $G_{\mathcal{I}}^+$ contains vertices $x_{i,j_1}, x_{i,j_2}, \overline{x_{i,j_3}}, \overline{x_{i,j_4}}$ which induce a 4-cycle and are all adjacent to y_j. Suppose that $y_j \in V_1$. Since V_1 is an independent set, we conclude that $x_{i,j_1}, x_{i,j_2}, \overline{x_{i,j_3}}, \overline{x_{i,j_4}} \in V_2 \cup V_3$. However, this contradicts the fact that $G_{\mathcal{I}}^+[V_2 \cup V_3]$ is a split graph. Thus $y_j \notin V_1$. By the same argument, $y_j \notin V_2$. This proves that $V_3 \supseteq \{y_1, \dots, y_m\}$. Furthermore, note that V_3 contains no $x_{i,j}$ or $\overline{x_{i,j}}$, since each has a non-neighbour in $\{y_1, \dots, y_m\}$ and V_3 is a clique. So $V_3 = \{y_1, \dots, y_m\}$ as promised.

This concludes the proof. □

Theorem 19. *The* STABLE-$\mathcal{M}_2^S$ *problem is NP-complete.*

PROOF. Once again we rephrase the problem as: given a graph G, decide if we can partition its vertex set into 3 independent sets V_1, V_2, V_3, such that the edges between V_2 and V_3 form the complement of a perfect matching. As before, we reduce from ONE-IN-THREE-3SAT.

Consider an instance $\mathcal{I}$ of the problem, namely the instance consists of m clauses $C_1, \dots, C_m$ containing variables $v_1, \dots, v_n$. Again, we may assume it satisfies the properties listed in Lemma 16. We define J_i to be the set of indices j such that v_i appears in C_j, and define $\overline{J_i}$ to be the set of indices j such that $\overline{v_i}$ appears in C_j.

For the instance $\mathcal{I}$, consider the graph $G_{\mathcal{I}}^+$ constructed in the proof of Theorem 18. Construct the graph $G_{\mathcal{I}}^*$ from $G_{\mathcal{I}}^+$ by removing all edges of the form $y_i y_j$ where $i, j \in \{1, \dots, m\}$ (effectively replacing the clique on $\{y_1, \dots, y_m\}$ by an independent set). All other edges remain the same.

We claim that $G_{\mathcal{I}}^*$ has a STABLE-$\mathcal{M}_2^S$ partition if and only if $\mathcal{I}$ has a satisfying truth assignment (as an instance of ONE-IN-THREE-3SAT).

For the forward direction, we note that, by the proof of Theorem 18, if $G_{\mathcal{I}}^+$ admits a STABLE-$\overline{\mathcal{M}_4^S}$ partition V_1, V_2, V_3, then $V_3 = \{y_1, \dots, y_m\}$. Thus, this is also a STABLE-$\mathcal{M}_2^S$ partition of $G_{\mathcal{I}}^*$. This proves that if $\mathcal{I}$ has a satisfying truth assignment, then $G_{\mathcal{I}}^*$ admits a STABLE-$\mathcal{M}_2^S$ partition.

Now suppose, conversely, that $G^*_{\mathcal{I}}$ admits a STABLE-$\mathcal{M}^S_2$ partition. In other words, $V(G^*_{\mathcal{I}})$ can be partitioned into three independent sets V_1, V_2, V_3, such that the edges between V_2 and V_3 form the complement of a perfect matching.

First, observe that if three vertices $a, b, c \in V_2 \cup V_3$ form an independent set then either all of them must be contained in V_2 or all of them must be contained in V_3. Indeed, suppose, without loss of generality, that $a, b \in V_2$ and $c \in V_3$, then c would have two non-neighbours in V_2, contradicting the fact that the edges between V_2 and V_3 form the complement of a perfect matching.

Next, we show that $y_j \in V_2$ for all $j \in \{1, \ldots, m\}$ or $y_j \in V_3$ for all $j \in \{1, \ldots, m\}$. By the above observation and Lemma 16, we need only show that $y_j \notin V_1$. Suppose, for contradiction, that $y_j \in V_1$. By Lemma 16, there must be vertices $x_{i_1,j_1}, x_{i_2,j_2}, x_{i_3,j_3}$ and $\overline{x_{i_1,j_4}}$ (with i_1, i_2, i_3 pairwise distinct), none of which correspond to literals in the clause C_j (i.e. $j \notin \{j_1, j_2, j_3, j_4\}$). Since they do not correspond to these literals, y_j must be adjacent to all of these vertices, so $x_{i_1,j_1}, x_{i_2,j_2}, x_{i_3,j_3}, \overline{x_{i_1,j_4}} \in V_2 \cup V_3$. But $x_{i_1,j_1}, x_{i_2,j_2}, x_{i_3,j_3}$ and $x_{i_2,j_2}, x_{i_3,j_3}, \overline{x_{i_1,j_4}}$ are both independent sets of size 3. Thus all four of these vertices must be members of the same set V_i where $i \in \{2, 3\}$. But x_{i_1,j_1} and $\overline{x_{i_1,j_4}}$ are adjacent, contradicting the fact that V_2 and V_3 are independent sets.

Hence, we may conclude, without loss of generality, that $\{y_1, \ldots, y_m\} \subseteq V_3$. Notice that, since each vertex $x_{i,j}$ or $\overline{x_{i,j}}$ corresponds to a unique occurrence of a literal in a unique clause in $\mathcal{I}$, every vertex not of the form y_j has a neighbour in V_3. Thus, since V_3 is an independent set, $V_3 = \{y_1, \ldots, y_m\}$. Finally, note that since V_1, V_2, V_3 is a STABLE-$\mathcal{M}^S_2$ partition for $G^*_{\mathcal{I}}$ and $V_3 = \{y_1, \ldots, y_m\}$, then by the construction of $G^*_{\mathcal{I}}$, it follows that V_1, V_2, V_3 must also be a STABLE-$\overline{\mathcal{M}^S_4}$-partition of $G^+_{\mathcal{I}}$. Thus, by the proof of Theorem 18, the instance $\mathcal{I}$ has a satisfying assignment.

This concludes the proof. □

6. Conclusion

We proved that the STABLE-Π problem is polynomial-time solvable for all subfactorial hereditary properties Π and for seven of the nine minimal factorial hereditary properties. For $\Pi = \mathcal{M}_1$, the problem is known to be NP-complete. This leaves one final open case, namely where Π is the class of chain graphs $\mathcal{M}_3$. Clarifying the complexity status of this exception is a challenging research problem.

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