New Results on Maximum Induced Matchings in Bipartite Graphs and Beyond

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Abstract

The maximum induced matching problem is known to be APX-hard in the class of bipartite graphs. Moreover, the problem is also intractable in this class from a parameterized point of view, i.e. it is W[1]-hard. In this paper, we reveal several classes of bipartite (and more general) graphs for which the problem admits fixed-parameter tractable algorithms. We also study the computational complexity of the problem for regular bipartite graphs and prove that the problem remains APX-hard even under this restriction. On the other hand, we show that for hypercubes (a proper subclass of regular bipartite graphs) the problem admits a simple solution.

Keywords: maximum induced matching, fixed-parameter tractability, regular bipartite graphs, hypercubes

1. Introduction

A *matching* in a graph is a subset of edges no two of which share of vertex. A matching is *induced* if no two vertices belonging to different edges of the matching are adjacent. In other words, an induced matching in a graph G is formed by the edges of a 1-regular induced subgraph of G. Induced matchings have also appeared under the name "strong matchings" [19]. Faudree et al. [14] were the first to study induced matchings in the context of bipartite graphs.

The MAXIMUM INDUCED MATCHING problem is that of finding an induced matching of maximum cardinality in a graph. We use $i\mu(G)$ to denote the

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maximum size of an induced matching in G. Due to various applications (see e.g. [20]), this problem has received much attention in recent years. It was originally introduced in [33], where it was called the "risk-free marriage problem".

From a computational point of view, finding an induced matching of maximum cardinality in a graph is an intractable problem in many respects. First of all, this problem is NP-hard, which was proved independently in [5] and [33]. Moreover, it remains NP-hard under substantial restrictions, for instance: for bipartite graphs of vertex degree at most 3 [27, 32], line graphs [26], planar graphs of vertex degree at most 4 [25] and even for cubic planar graphs [11]. On the other hand, polynomial-time algorithms for this problem have been developed for weakly chordal graphs [7], AT-free graphs [8], circular arc graphs [19], graphs of bounded clique-width [26] and some other classes of graphs [3, 4, 6, 20, 27].

The problem also remains intractable in terms of finding approximation algorithms. In particular, in [11] the problem was shown to be APX-complete in cubic graphs and in bipartite graphs where the minimum degree is $2s$ and the maximum degree is $3s$, for any positive integer s. In [30] it was shown that the problem is not approximable within a factor of $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$. Some other inapproximability results can be found in [9] and in [11], where some explicit lower bounds for the approximation ratio are given. Conversely, a subclass of bipartite graphs has been found in which the problem admits a polynomial-time approximation scheme [13]. For all notions related to approximation theory not defined in this paper, the reader is referred to [1].

The problem was also shown to be intractable from a parameterized point of view. More precisely, it is W[1]-hard in general [29] and even when restricted to bipartite graphs [28]. In Section 3 we reveal a number of graph classes, including subclasses of bipartite graphs, where the problem admits fixed-parameter tractable algorithms.

Much attention has been given to the problem in regular graphs (see e.g. [2, 9, 11, 12, 21]). We contribute to this topic in two different ways. In Section 4 we show that the problem is APX-complete in k-regular bipartite graphs, for any $k \geq 3$, which was previously unknown, despite the fact that finding a maximum induced matching was known to be APX-hard both for regular graphs and for bipartite graphs. In contrast to this negative result we show that the problem admits a simple solution for hypercubes (a proper subclass of regular bipartite graphs).

2. Preliminaries

All graphs in this paper are undirected, without loops or multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. If v is a vertex of G, then $N(v)$ is the *neighbourhood* of v (i.e. the set of vertices adjacent to v). The *degree* of v is the size of its neighbourhood.

For a subset $U \subseteq V(G)$ we let $N(U) = \cup_{v \in U} N(v)$ denote the neighbourhood of U and $G[U]$ denote the subgraph of G induced by U. Note that if two distinct vertices have the same neighbourhood, they must be nonadjacent. A *module* in a graph is a set X of vertices, such that every vertex not in X is adjacent to either all or none of the vertices in X. A module is *trivial* if it contains either all vertices of G, exactly one vertex of G or if it is empty, otherwise it is *non-trivial.* A graph is said to be *prime* if every module in the graph is trivial.

If a graph G does not contain an induced subgraph isomorphic to a graph H, then we say that G is *H-free.* Two distinct edges are said to be *linked* if they are joined by an edge or if they have a common end-vertex. K_n, C_n and P_n denote the complete graph, the chordless cycle and the chordless path on n vertices, respectively. $K_{n,m}$ is the complete bipartite graph, also known as a bi-clique, with parts of size n and m. If $G = (U, V, E)$ is a bipartite graph, the *bipartite complement* of G is the bipartite graph $(U, V, (U \times V) \setminus E)$.

The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Creating a new vertex v with label i (denoted by $i(v)$).
(ii) Taking the disjoint union of two labelled graphs G and H (denoted by $G \oplus H$).
(iii) Joining each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$).
(iv) Renaming label i to j (denoted by $\rho_{i\to j}$).

Every graph can be defined by an algebraic expression using these four operations. For instance, an induced path on five consecutive vertices a, b, c, d, e has clique-width equal to 3 and it can be defined as follows:

$$\eta_{3,2}(3(e) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))))))$$

3. Parameterized complexity of the problem

In parameterized complexity theory, an instance of a graph problem is a pair (G, k), where G is a graph and k is a parameter assigning a natural number to each graph. A parameterized problem is *fixed-parameter tractable* if it can be solved in $f(k)n^{O(1)}$ time, where n is the number of vertices in G and $f(k)$ is a computable function depending only on the value of the parameter k (see [10, 15] for more information on parameterized complexity theory). An fpt algorithm is one that solves the parameterized problem in $f(k)n^{O(1)}$ time.

We study the following parameterization of the MAXIMUM INDUCED MATCHING problem:

k-INDUCED MATCHING
Instance: A graph G and a positive integer k.
Parameter: k.
Problem: Decide whether G has an induced matching of size at least k and find such a matching if it exists. If no such matching exists, find an induced matching of size $i\mu(G)$ instead.

The parameterized complexity of the k-INDUCED MATCHING problem was studied in [29] and [28]. In particular, in [29] it was shown that the problem is W[1]-hard and hence unlikely to be fixed-parameter tractable when parameterized by the solution size. In [28], this result was strengthened by showing that the problem is also W[1]-hard when restricted to bipartite graphs. On the other hand, in [28] the problem was shown to be fixed-parameter tractable in several classes such as planar graphs (see also [24] for a better fpt algorithm for planar graphs with maximum degree 3), graphs of bounded degree and the class of (C_3, C_4, C_5)-free graphs. Observe that the latter class includes, in particular, all C_4-free bipartite graphs, where the problem is known to be NP-hard [27]. We generalise the fixed-parameter tractability of the problem in the class of C_4-free bipartite graphs in two different ways. First, in Section 3.1 we prove fixed-parameter tractability of the problem in the class of $(K_s, K_{t,t})$-free graphs for arbitrary values of s and t, which also generalises the results for (C_3, C_4, C_5)-free graphs, planar graphs and graphs of bounded degree. Second, in Section 3.2 we present an fpt algorithm for so-called A-free bipartite graphs.

Now, let us note that in any graph, if two vertices x and y have the same neighbourhood, at most one of them can be the endpoint of an edge in any induced matching. Consequently, for computing a maximum induced matching we can first look for every pair (x, y) of vertices with the same neighbourhood and arbitrarily delete one of them. This can be done in polynomial time.

Remark 1. *If two vertices in a graph have the same neighbourhood, we can arbitrarily delete one of them and the size of the maximum induced matching will be unchanged.*

Modules in bipartite graphs have the following property.

Remark 2. *In a connected bipartite graph G, every non-trivial module must be an independent set.*

Indeed, if a non-trivial module X of vertices in G contains two vertices y, y' that are adjacent, then they must be in different parts of the bipartition of G. Since X is a non-trivial module and the graph G is connected, there must be a vertex z outside of X with a neighbour $x \in X$. This vertex z can be connected to at most one of y and y', since the graph is bipartite, contradicting the claim that X is a module.

From this remark, it follows that in any non-trivial module X of a connected bipartite graph G, every vertex of X must have the same neighbourhood. This means that if G is a bipartite graph and no two vertices of G have the same neighbourhood, then every component of G is prime.

Since the problem can be solved independently on each component of a graph, we can therefore draw the following conclusion:

Remark 3. *The k-*Induced Matching* problem is fixed-parameter tractable in a hereditary class of bipartite graphs $\mathcal{C}$ if and only if it is fixed-parameter tractable in the class of prime graphs in $\mathcal{C}$.*

3.1. An fpt algorithm for $(K_s, K_{t,t})$-free graphs

We denote an induced matching with p edges by M_p. Also, we let $R(s,t)$ and $Rb(s,t)$ denote the non-symmetric Ramsey and non-symmetric bipartite Ramsey numbers, respectively. In other words, $R(s,t)$ is the minimum number such that if G is a graph on at least $R(s,t)$ vertices, then either G contains K_s as an induced subgraph or the complement of G contains K_t as an induced subgraph (i.e. G contains an independent set of size t). Similarly, $Rb(s,t)$ is the minimum number such that if G is a bipartite graph with at least $Rb(s,t)$ vertices in each part then either G contains $K_{s,s}$ as an induced subgraph or the bipartite complement of G contains $K_{t,t}$ as an induced subgraph.

Lemma 1. *For any natural numbers t and p, there is a number $N(t,p)$ such that every bipartite graph with a matching of size at least $N(t,p)$ contains either a bi-clique $K_{t,t}$ or an induced matching M_p.*

Proof. For $p = 1$ and arbitrary t, we can define $N(t,p) = 1$. Now, for each fixed t, we prove the lemma by induction on p. Without loss of generality, we prove it only for values of the form $p = 2^s$. Suppose we have shown the lemma for $p = 2^s$ for some $s \geq 0$. Let us now show that it is sufficient to set $N(t,2p) = Rb(t, Rb(t, N(t,p)))$.

Consider a graph G with a matching of size at least $Rb(t, Rb(t, N(t,p)))$. Without loss of generality, we may assume that G contains no vertices outside of this matching. We also assume that G does not contain an induced $K_{t,t}$, since otherwise we are done. Then G must contain the bipartite complement of a $K_{Rb(t,N(t,p)),Rb(t,N(t,p))}$ with vertex classes, say, A and B. Now let C and D consist of the vertices matched to vertices in A and B respectively in the original matching in G.

Note that A, B, C, D are pairwise disjoint. $G[A \cup C]$ and $G[B \cup D]$ now each contain a matching of size $Rb(t, N(t,p))$. There are no edges between A and B. However there may exist edges between C and D. By our assumption, $G[C \cup D]$ is $K_{t,t}$-free, therefore it must contain the bipartite complement of $K_{N(t,p),N(t,p)}$, with vertex sets $C' \subset C$, $D' \subset D$. Let $A' \subset A$ and $B' \subset B$ be the set of vertices matched to C' and D' respectively in the original matching in G. Now there are no edges in $G[A' \cup B']$ and none in $G[C' \cup D']$, but $G[A' \cup C']$ and $G[B' \cup D']$ both contain a matching of size $N(t,p)$. Since G is $K_{t,t}$-free, by the induction hypothesis, we conclude that they both contain an induced M_p. Putting these together we find that G contains an induced M_{2p}. □

Lemma 2. *For any natural numbers s, t and p, there is a number $N'(s,t,p)$ such that every graph with a matching of size at least $N'(s,t,p)$ contains either a clique K_s, an induced bi-clique $K_{t,t}$ or an induced matching M_p.*

Proof. We will show that setting $N'(s,t,p) = R(s, R(s, N(t,p)))$ is sufficient. Indeed, suppose G is a $(K_s, K_{t,t})$-free graph with a matching of size $R(s, R(s, N(t,p)))$. G is K_s-free, so it must contain an independent set A of size $R(s, N(t,p))$. Let B be the set of vertices matched to A. Since $G[B]$ is

K_s-free, it must contain an independent set B' of size $N(t,p)$. Let A' be the set of vertices matched to B'. Now $G[A' \cup B']$ is a bipartite graph with a matching of size $N(t,p)$. By Lemma 1, $G[A' \cup B']$ contains an induced matching M_p. □

Theorem 1. *For each fixed s and t, the k-INDUCED MATCHING problem is fixed-parameter tractable in the class of $(K_s, K_{t,t})$-free graphs.*

PROOF. Fix s and t and let G be a $(K_s, K_{t,t})$-free graph with n vertices. We will show that the problem of determining whether G has an induced matching of size k can be solved in time $f(k)p(n)$, where $f(k)$ is a function of k only and $p(n)$ is a polynomial in n independent of k.

Let M be a maximal (with respect to set inclusion) matching in G. Clearly, such a matching can be found in polynomial time. If M is of size at least $N'(s,t,k)$, then by Lemma 2, G has an induced matching of size k. To find such a matching, we can restrict ourselves to $N'(s,t,k)$ edges of M. This reduces the problem to a subgraph G induced by $2N'(s,t,k)$ vertices, at which point the problem can be solved in $O(N'(s,t,k)^{2k}k^2)$ time, i.e. independent of n.

If M contains less than $N'(s,t,k)$ edges we proceed as follows. Let V_M be the set of vertices which are endpoints of edges in M. If $xy \in E(G)$, then either $x \in V_M$ or $y \in V_M$ (otherwise M would not be maximal). By Remark 1, we may assume no two vertices have the same neighbourhood.

So we are now reduced to a graph in which the neighbourhood of every vertex $v \in V \setminus V_M$ is contained in V_M and no two vertices have the same neighbourhood. Thus the graph contains at most $2^{2N'(s,t,k)} + 2N'(s,t,k)$ vertices and we can solve the k-induced matching problem in $O((2^{2N'(s,t,k)} + 2N'(s,t,k))^{2k}k^2)$ time, which is independent of n.

Summarising, we conclude that the problem is fixed-parameter tractable in the class of $(K_s, K_{t,t})$-free graphs. □

Corollary 1. *For each fixed t, the k-INDUCED MATCHING problem is fixed-parameter tractable in the class of $(K_{t,t})$-free bipartite graphs.*

3.2. An fpt algorithm for A-free bipartite graphs

In this section, we develop an fpt algorithm for the k-INDUCED MATCHING problem in the class of A-free bipartite graphs, where A is the graph represented in Figure 1. Since A contains a C_4, the class of A-free bipartite graphs extends C_4-free bipartite graphs.

Clearly, the problem can be reduced to connected graphs. More importantly, by Remark 3, the problem can be reduced to bipartite graphs which are prime.

Let G be a connected prime A-free bipartite graph. If G contains no C_4, we apply the fpt algorithm from the previous section (since $C_4 = K_{2,2}$). If G contains a C_4, we apply the following structural characterisation of G.

Lemma 3. *Let $G = (U, V, E)$ be a connected prime A-free bipartite graph, containing a C_4. Then the vertices of G can be partitioned into four subsets U_0, U_1, V_0, V_1 in such way that $U_0 \cup V_0$ and $U_1 \cup V_1$ induce complete bipartite graphs, while $U_0 \cup V_1$ and $U_1 \cup V_0$ induce P_6-free bipartite graphs.*

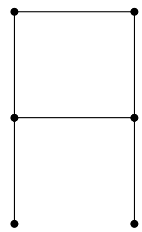

Figure 1: The graph A

Proof. Consider a connected prime A-free bipartite graph $G = (U, V, E)$ containing a C_4 and let $H = G[U_0 \cup V_0]$ be a maximal (with respect to inclusion) complete bipartite subgraph containing this C_4. Also, for $i \geq 1$, let U_i and V_i be the set of vertices in U and V at distance i from $U_0 \cup V_0$. Then $U_1 \cup V_1$ induces a complete bipartite graph. Indeed, assume for contradiction that a vertex $a \in U_1$ is not adjacent to a vertex $x \in V_1$. By definition, a must have a neighbour $b \in V_0$ and a non-neighbour $c \in V_0$ (since otherwise H would not be a maximal complete bipartite subgraph containing the initial C_4). Similarly, x must have a neighbour $y \in U_0$ and a non-neighbour $z \in U_0$. But then a, b, c, x, y, z induce an A.

Notice that each of U_0 and V_0 contains at least 2 vertices, which together with the primality of G implies that U_1 is not empty and V_1 is not empty, since otherwise any two vertices of U_0 or V_0 would have the same neighbourhood. As a result, we can conclude that for all $i > 1$ the sets U_i and V_i are empty. Indeed, assume for contradiction that U_2 contains a vertex a. Then by definition it must have a neighbour $x \in V_1$, while x must have a neighbour c in U_0. Since U_1 is not empty, we may consider an arbitrary vertex $b \in U_1$, an arbitrary neighbour $y \in V_0$ of b and an arbitrary non-neighbour (which exists due to maximality of H) $z \in V_0$ of b. But then a, b, c, x, y, z induce an A. This contradiction shows that U_2 is empty, and by symmetry we conclude that V_2 is empty.

Assume now that $G[U_1 \cup V_0]$ contains an induced $P_6 = (x_1, x_2, x_3, x_4, x_5, x_6)$ with $x_1, x_3, x_5 \in U_1$ and $x_2, x_4, x_6 \in V_0$, and let a be an arbitrary vertex in U_0. Then $a, x_1, x_2, x_3, x_4, x_6$ induce an A in G. This contradiction proves that $G[U_1 \cup V_0]$ is P_6-free, and by symmetry we conclude that $G[U_0 \cup V_1]$ is P_6-free. □

Corollary 2. *The clique-width of A-free connected prime bipartite graphs containing a C_4 is bounded by a constant.*

Proof. It is known (see e.g. [17]) that the clique-width of bipartite graphs in a given hereditary class is bounded if and only if it is bounded for graphs in this class which are connected and whose bipartite complement is also connected.

Let $\mathcal{C}$ denote the class of connected prime A-free bipartite graphs containing a C_4. Let G be an induced subgraph of a graph in $\mathcal{C}$. By Lemma 3, G either has disconnected bipartite complement or it is a P_6-free bipartite graph. However, P_6-free bipartite graphs are known to have bounded clique-width (see e.g. [18]), which completes the proof. □

For graphs of bounded clique-width, the MAXIMUM INDUCED MATCHING problem is known to be solvable in polynomial-time [26]. This fact together with Corollary 2 and an fpt algorithm for the k-INDUCED MATCHING problem in the class of C_4-free graphs, leads to the following conclusion.

Theorem 2. *The k-INDUCED MATCHING problem is fixed-parameter tractable in the class of A-free bipartite graphs.*

4. Regular bipartite graphs

As we mentioned in the introduction, much attention has been given to the problem in regular graphs. In particular, [12] and [2] study the problem in random regular graphs, while [9, 11, 21] study approximability of the problem in regular graphs. This interest in regular graphs is partly due to the fact that the problem remains NP-hard in this class. It is also known that the problem is NP-hard for bipartite graphs of bounded degree. However, the complexity of the problem in regular bipartite graphs was unknown. We answer this question negatively by showing that the problem is APX-complete in k-regular bipartite graphs for any $k \geq 3$, implying that there is a constant $c > 1$ such that it is NP-hard to approximate the problem to within a factor of c. An interesting subclass of regular bipartite graphs is the class of hypercubes. We show that in this case the problem admits a simple solution.

4.1. APX-completeness

The proof follows from an approximation preserving reduction. We use the L-reduction as defined in [31]. Let P be a maximization problem. For every instance x of P, and every solution y of x, let $c_P(x, y)$ be the *cost* of the solution y. Let $opt_P(x)$ be the cost of an optimal solution. If $c \geq 1$ is a constant and there is a polynomial time algorithm that computes a solution $y(x)$, such that $\forall x, c_P(x, y(x)) \geq \frac{1}{c} opt_P(x)$, then the algorithm is said to approximate P to within a *ratio* of c. If this holds for a constant $c > 1$, then P is said to be constant-factor approximable and it belongs to the class APX. If, for any positive ε, P has a polynomial time algorithm guaranteeing a ratio of $1 + \varepsilon$, then we say that P has a polynomial-time approximation scheme (PTAS).

Definition 1. *Let P and Q be two maximization problems. An* L-reduction *from P to Q is a four-tuple $(t_1, t_2, \alpha, \beta)$, where t_1 and t_2 are polynomial time computable functions and α and β are positive constants with the following properties:*

(a) *t_1 maps instances of P to instances of Q and for every instance x of P, $opt_Q(t_1(x)) \leq \alpha opt_P(x)$.*

(b) *For every instance x of P, t_2 maps pairs $(t_1(x), y')$ (where y' is a solution of $t_1(x)$) to a solution y of x so that* $|opt_P(x) - c_P(x, t_2(t_1(x), y'))| \leq \beta |opt_Q(t_1(x)) - c_Q(t_1(x), y')|$.

As shown in [31], if P and Q are maximization problems and there is an L-reduction from P to Q then if Q has a PTAS, P must also have a PTAS. Conversely, the definition of APX-hardness implies that if P is APX-complete, then Q is APX-hard. If furthermore Q is in APX, then it is APX-complete.

For any finite set D of positive integers, we say a graph G is a *D-graph* if D is the set of vertex degrees in G. For example a $\{k\}$-graph is a non-empty k-regular graph.

Theorem 3. *Let D be a finite set of positive integers such that* $\max_{d \in D} d \geq 3$, *then* MAXIMUM INDUCED MATCHING *is APX-complete in the class of bipartite D-graphs. In particular, it is APX-complete in the class of k-regular bipartite graphs for any $k \geq 3$.*

PROOF. The MAXIMUM INDUCED MATCHING problem is known to be approximable to within a constant factor in k-regular graphs [34]; so it remains to show it is APX-hard.

For any fixed $k \geq 3$, we define the gadget $H_k = (V_k, E_k)$ (see Figure 2) as follows:

The set of vertices is defined by $V_k = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \cup L_6$ with

$$L_1 = \{1_1, \ldots, 1_k\}, \quad L_2 = \{2_1, \ldots, 2_k\}, \quad L_3 = \{3_1, \ldots, 3_{k(k-1)}\},$$
$$L_4 = \{4_1, \ldots, 4_{k(k-1)}\}, \quad L_5 = \{5_1, \ldots, 5_{(k-1)^2}\}, \quad L_6 = \{6_1, \ldots, 6_{(k-1)(k-2)}\}.$$

For $i = 1, \ldots, k-1$, we denote

$$S_3^i = \{3_{(i-1)k+1}, \ldots, 3_{ik}\}, \quad S_4^i = \{4_{(i-1)k+1}, \ldots, 4_{ik}\},$$
$$S_5^i = \{5_{(i-1)(k-1)+1}, \ldots, 5_{i(k-1)}\}, \quad S_6^i = \{6_{(i-1)(k-2)+1}, \ldots, 6_{i(k-2)}\}.$$

So $|S_3^i| = |S_4^i| = k$, $|S_5^i| = k-1$ and $|S_6^i| = k-2$.

The set of edges E_k is defined as follows:

(1) $L_1 \cup L_2$ induces a matching of size k: $(1_i, 2_i) \in E_k, i = 1, \ldots, k$.

(2) $(2_i, 3_{(i-1)(k-1)+j}) \in E_k, i = 1, \ldots, k, j = 1, \ldots, k-1$.

(3) $L_3 \cup L_4$ induces a matching: $(3_i, 4_i) \in E_k, i = 1, \ldots, k(k-1)$.

(4) For every $i = 1, \ldots, k-1$, S_4^i and S_5^i induce a $K_{k,k-1}$.

(5) For every $i = 1, \ldots, k-1$, S_3^i and S_6^i induce a $K_{k,k-2}$.

Note that every vertex of L_1 is of degree 1 in H_k while the other vertices are of degree k. Note also that $\forall i \in \{1, \ldots, k-1\}, N(S_3^i) \cap L_2 = \{2_i, 2_{i+1}\}$.

For any graph $G = (V, E)$ and any set of k vertices $S = \{v_1, \ldots, v_k\} \subset V$, we define the graph $G \cup_S H_k$ obtained by adding an H_k to G and identifying

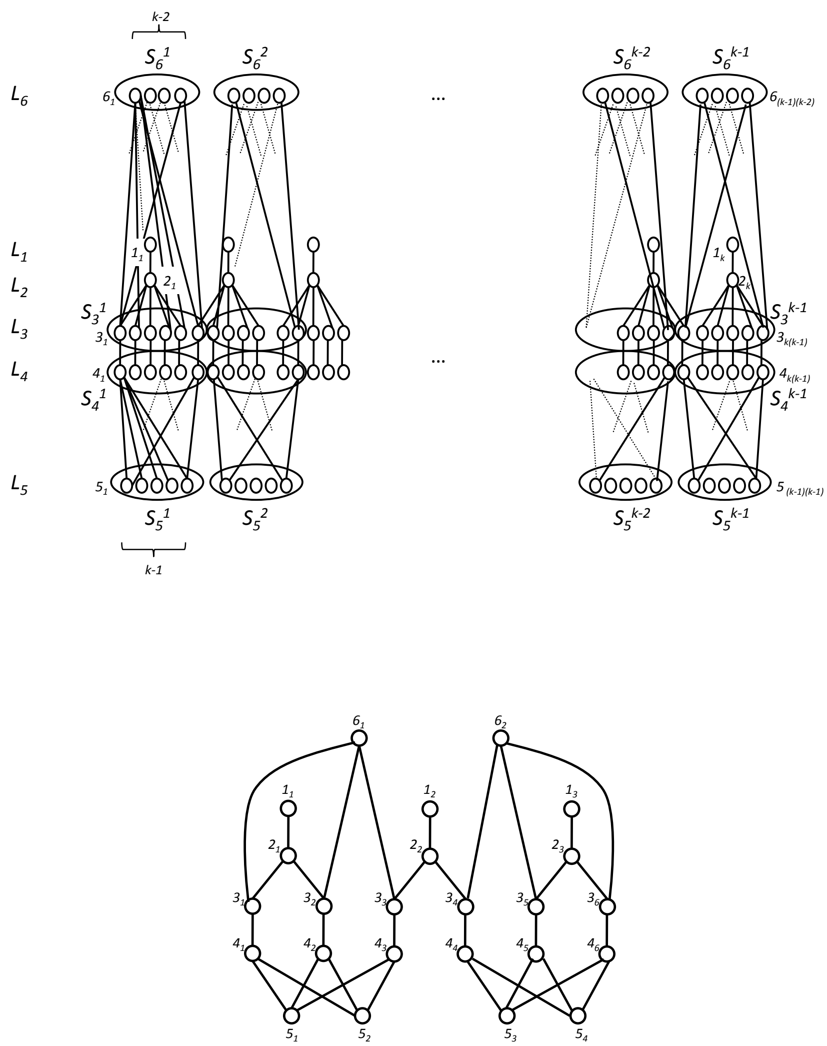

Figure 2: Gadget H_k and example for $k = 3$.

L_1 and S. More formally its set of vertices is $V \cup L_2 \cup L_3 \cup L_4 \cup L_5 \cup L_6$ and $(G \cup_S H_k)[V] = G$ and $(G \cup_S H_k)[S \cup L_2 \cup L_3 \cup L_4 \cup L_5 \cup L_6] = H_k$. For any two graphs $G = (V, E), G' = (V', E')$ we denote $G \cup G' = (V \cup V', E \cup E')$.

Lemma 4. *For any $k \geq 3$, $\{(3_i, 4_i), i = 1, \ldots, k(k-1)\}$ is a maximum induced matching of H_k.*

Proof. Note first that, since vertices 1_i, $i = 1, \ldots, k$ are of degree 1 in H_k, for any induced matching M of H_k containing an edge $(2_i, 3_{(i-1)(k-1)+j})$, with $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, k-1\}$, $M \setminus \{(2_i, 3_{(i-1)(k-1)+j})\} \cup \{(1_i, 2_i)\}$ is also an induced matching. Consequently, without loss of generality we can restrict ourselves to the case where M does not contain any edge $(u, v), u \in L_2, v \in L_3$. For every $i = 1, \ldots, k-1$, we let $M_i = M \cap [\{(1_i, 2_i), (1_{i+1}, 2_{i+1})\} \cup \{(u, v), u \in S_3^i, v \in S_6^i\} \cup \{(u, v), u \in S_4^i, v \in S_5^i\}]$. Note that $|M_i| \leq 3$. Since edges $(u, v), u \in S_3^i, v \in S_4^i$ constitute an induced matching and are only linked, in M, to edges belonging to M_i, $M' = M \setminus M_i \cup \{(u, v), u \in S_3^i, v \in S_4^i\}$ is also an induced matching. Moreover, if M contains an edge $(u, v), u \in S_3^i, v \in S_4^i$, then it does not contain any edge $(u', v'), u' \in S_3^i, v' \in S_6^i$ or any edge $(u'', v''), u'' \in S_4^i, v'' \in S_5^i$. So $|M'| \geq |M|$ and consequently, there is a maximum induced matching of H_k containing edges $(u, v), u \in L_3, v \in L_4$. Since this matching is maximal, Lemma 4 follows. □

A direct consequence of Lemma 4 is that there is a maximum induced matching in $G \cup_S H_k$ containing $\{(3_i, 4_i), i = 1, \ldots, k(k-1)\}$ and consequently

$$i\mu(G \cup_S H_k) = i\mu(G) + k(k-1) \tag{1}$$

Moreover, if G is bipartite and S is monochromatic for a fixed 2-colouring of G, then $G \cup_S H_k$ is also bipartite.

We can now describe the reduction. Let $G = (V, E)$ be any bipartite D-graph. Let us first note that for any positive integer d, $G \cup K_{d,d}$ is a bipartite $(D \cup \{d\})$-graph and

$$i\mu(G \cup K_{d,d}) = i\mu(G) + 1 \tag{2}$$

On the other hand, for any $d \in D$, let $u_d^1, \ldots, u_d^p$, $p \geq 1$ denote the vertices of degree d. Let $k \geq 3$. We consider k copies of G denoted by $G_1, \ldots, G_k$ and for any vertex $v \in V(G)$ we let $S(v)$ denote the set of copies of v in $G_1, \ldots, G_k$ (so $|S(v)| = k$). We then define:

$$T_d^k(G) = (G_1 \cup \ldots \cup G_k) \cup_{S(u_d^1)} H_k \ldots \cup_{S(u_d^p)} H_k$$

Using relation (1) we immediately obtain:

$$i\mu(T_d^k(G)) = ki\mu(G) + pk(k-1) \tag{3}$$

It is also straightforward to verify that, if $k \in D \cup \{d+1\}, d \neq k$, then $T_d^k(G)$ is a bipartite $(D \setminus \{d\} \cup \{d+1\})$-graph. Since $G \cup K_{d,d}$ and $T_d^k(G)$

can be performed in polynomial time, relations (2) and (3) imply that the related reduction preserves polynomial approximation schema. The first of these is an L-reduction with $\alpha = 2, \beta = 1$ (since $i\mu(G) \geq 1$ follows immediately from the fact that G is a D-graph) and the second is an L-reduction with $\alpha = k(1 + 2(2\Delta(\Delta - 1) + 1)(k - 1))$ and $\beta = 1/k$, where $\Delta = \max(D)$.

Indeed, observe that since the degree in G is bounded above by Δ, any edge in G is linked to at most $2\Delta(\Delta - 1)$ edges. Since D is made up of positive integers, the minimum degree in G is at least 1, so there are at least $\frac{n}{2}$ edges in G. Thus $i\mu(G) \geq \frac{n}{2(2\Delta(\Delta-1)+1)}$. Note that $p \leq n$ for this transformation. This means that

$$\begin{aligned} i\mu(T_d^k(G)) &= ki\mu(G) + pk(k-1) \\ &\leq ki\mu(G) + nk(k-1) \\ &\leq k(1 + 2(2\Delta(\Delta - 1) + 1)(k - 1))i\mu(G) \end{aligned}$$

as required.

Consequently if the MAXIMUM INDUCED MATCHING problem is APX-complete in bipartite D-graphs, then for any positive integer d it is also APX-complete in bipartite $(D \cup \{d\})$-graphs and, using the transformation T_d^{d+1} for any $d \in D, d \geq 2$, it is also APX-complete for bipartite $(D \setminus \{d\} \cup \{d+1\})$-graphs.

The problem is shown to be APX-complete for bipartite $\{2, 3\}$-graphs [11]. (More precisely, for any $\varepsilon > 0$, the problem of approximating MAXIMUM INDUCED MATCHING within a factor of $\frac{9570}{9569} - \varepsilon$ is NP-hard for graphs in this class.) Then, using the above remarks successively for $d = 2, 3, \ldots$ we deduce that it is APX-complete in bipartite $\{3\}$-, $\{4\}$-, ..., $\{k\}$-graphs for any $k \geq 3$ and consequently, that it is APX-complete in bipartite D-graphs for any finite D with at least one element $k \geq 3$. □

4.2. Hypercubes

The n-dimensional hypercube Q_n is the graph with vertex set $\{0, 1\}^n$, where two vertices are adjacent if and only if they differ in precisely 1 coordinate. Thus, the number of vertices in Q_n is 2^n and every vertex has degree n, i.e. Q_n is a regular graph. It is not difficult to see that Q_n is also a bipartite graph, since vertices of the same parity are necessarily non-adjacent. Hypercubes enjoy many more nice graph-theoretic properties (see e.g. [22]). Nonetheless, there are algorithmic problems on hypercubes for which no efficient algorithms are known. This is the case, for instance, for the crossing number and the size of a smallest maximal matching for which only bounds are available (see e.g. [16, 23]) and no efficient algorithms to compute the respective numbers are known. However, this is not the case for the size of a maximum induced matching. Below we present a simple formula for this number. The proof is constructive and exhibits an easy way of finding an induced matching of this size.

Theorem 4. *For $n \geq 2$, a maximum induced matching in the hypercube Q_n has 2^{n-2} edges.*

PROOF. Consider the set of vertices $M = \{(x_1, \ldots, x_n),\ x_2 \equiv x_3 + \ldots + x_n \mod(2)\}$. M contains 2^{n-1} points. Note that $x, y \in M$ are neighbours in Q_n if and only if they differ in the x_1 coordinate, but do not differ in any of the other coordinates. Therefore $Q_n[M]$ is an induced matching in Q_n which contains 2^{n-2} edges.

Clearly this is optimal if $n = 2$. For $n > 2$, fix $x_1, \ldots, x_{n-2} \in \{0, 1\}$ and consider the points

$$(x_1, \ldots, x_{n-2}, 0, 0), (x_1, \ldots, x_{n-2}, 0, 1), (x_1, \ldots, x_{n-2}, 1, 0), (x_1, \ldots, x_{n-2}, 1, 1).$$

At most 2 of these points can be endpoints of edges in any induced matching in Q_n. Therefore in any induced matching in Q_n, at most half of the vertices of Q_n can be endpoints of edges in the matching. So the number of edges in such a matching is at most 2^{n-2}. □

5. Future Work

In Theorem 3, we showed that the MAXIMUM INDUCED MATCHING problem is APX-complete for k-regular bipartite graphs. A few results with a constant multiplicative error have been established for regular graphs [21, 11, 34], but no such results exist specifically for the bipartite case. An interesting direction for future research is to investigate the approximation of the MAXIMUM INDUCED MATCHING problem in regular bipartite graphs and in particular to see whether approximation results in regular graphs can be widely improved in the bipartite case.

In [13], the problem was found to have a polynomial-time approximation scheme in the class of bipartite graphs with the property that all the vertices in one part of the partition have degree 2 and all the vertices in the other part have degree 3. Since the problem is APX-complete in the class of bipartite {2,3}-graphs, interesting approximation results may emerge from the study of bipartite instances where all vertices in one part have degree 2 and all vertices in the other part have some other degree d.

Since the problem, is fixed-parameter tractable in the class of $(K_s, K_{t,t})$-free graphs a natural question that arises is whether one can find a small problem kernel in these classes (see [10] for the technical definition of kernel). Algorithms to obtain such kernels have been found in some subclasses of $(K_s, K_{t,t})$-free graphs (see e.g. [28]).

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