Colouring of Graphs with Ramsey-Type Forbidden Subgraphs[*]

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Abstract. A colouring of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ if $uv \in E$; if $|c(V)| \leq k$ then c is a k-colouring. The Colouring problem is that of testing whether a given graph has a k-colouring for some given integer k. If a graph contains no induced subgraph isomorphic to any graph in some family $\mathcal{H}$, then it is called $\mathcal{H}$-free. The complexity of Colouring for $\mathcal{H}$-free graphs with $|\mathcal{H}| = 1$ has been completely classified. When $|\mathcal{H}| = 2$, the classification is still wide open, although many partial results are known. We continue this line of research and forbid induced subgraphs $\{H_1, H_2\}$, where we allow H_1 to have a single edge and H_2 to have a single non-edge. Instead of showing only polynomial-time solvability, we prove that Colouring on such graphs is fixed-parameter tractable when parameterized by $|H_1| + |H_2|$. As a byproduct, we obtain the same result both for the problem of determining a maximum independent set and for the problem of determining a maximum clique.

1 Introduction

Graph colouring involves the labelling of the vertices of some given graph by integers called colours such that no two adjacent vertices receive the same colour. The Colouring problem is that of deciding whether or not a graph can be coloured with at most k colours for some given integer k. Because Colouring is NP-complete for any fixed $k \geq 3$, its computational complexity has been widely studied for special graph classes, see for example the surveys of Randerath and Schiermeyer [23] and Tuza [26]. In this paper, we consider the Colouring problem for graphs characterized by two forbidden induced subgraphs. Before we summarize related results and explain our new results, we first state the necessary terminology.

1.1 Basic Terminology

We only consider finite undirected graphs without loops or multiple edges. We refer to the textbook of Diestel [10] for any undefined graph terminology. Let $G = (V, E)$ be a graph. A *colouring* of G is a mapping $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. We call $c(u)$ the *colour* of u and $\{u \in V \mid c(u) = i\}$ for some $i \geq 1$ a *colour class* of c. A k-*colouring* of G is a colouring c of G with $1 \leq c(u) \leq k$ for all $u \in V$. The smallest integer k for which G has a k-colouring is called the *chromatic number* of G, denoted $\chi(G)$; a $\chi(G)$-colouring is said to be *optimal*. The k-COLOURING problem is that of deciding whether a given graph admits a k-colouring. Here, k is *fixed*, that is, not part of the input. If k is part of the input, then we denote the problem as COLOURING.

Let $G = (V, E)$ be a graph. A graph H is an *induced subgraph* of G if H can be obtained from G by deleting zero or more vertices. In this case we write $H \subseteq_i G$. For a set $S \subseteq V$, we let $G[S] = (S, \{uv \in E \mid u, v \in S\})$ denote the subgraph of G *induced by* S. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. We say that G is $(H_1, \ldots, H_p)$-*free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write that G is H_1-free instead of (H_1)-free.

The *complement* of a graph $G = (V, E)$, denoted by $\overline{G}$, has vertex set V and an edge between two distinct vertices if and only if these vertices are not adjacent in G. The *disjoint union* of two graphs G and H with $V(G) \cap V(H) = \emptyset$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and we denote this by $G + H$. We denote the disjoint union of r copies of G by rG.

For $r \geq 1$, the graph P_r denotes the *path* on r vertices, that is, $V(P_r) = \{u_1, \ldots, u_r\}$ and $E(P_r) = \{u_i u_{i+1} \mid 1 \leq i \leq r - 1\}$. Adding the edge $u_1 u_r$ to this graph yields the *cycle* on r vertices, denoted C_r. The graph sP_1 denotes the *independent set* on s vertices. The INDEPENDENT SET problem is that of testing whether a given graph has an independent set of size at least k for some given integer k. The graph K_t denotes the *complete* graph on t vertices, that is, $V(K_t) = \{u_1, \ldots, u_t\}$ and $E(K_t) = \{u_i u_j \mid 1 \leq i < j \leq t\}$. The vertex set of a complete graph is called a *clique*. The CLIQUE problem is that of testing whether a given graph has a clique of size at least k for some integer k. The graph $K_t - e$ denotes the graph obtained from K_t after removing exactly one edge.

1.2 Our Results

We show fixed-parameter tractability results for three problems, namely for COLOURING, INDEPENDENT SET and CLIQUE, restricted to $(sP_1 + P_2, K_t - e)$-free graphs. In parameterized complexity theory, the problem input consists of a pair (I, p), where I is the problem instance and p is the parameter. A problem is *fixed-parameter tractable* (fpt) with parameter p if it can be solved in time $f(p) \cdot |I|^{O(1)}$ for some function f that only depends on p. In our case, a natural parameter is $s + t$. In Section 2 we show that COLOURING is fixed-parameter tractable with parameter $s + t$ when restricted to $(sP_1 + P_2, K_t - e)$-free graphs, that is, can be solved in time $f(s + t)(n + k)^{O(1)}$ for some function f that only depends on $s + t$. In the same section, we show that the INDEPENDENT SET and

CLIQUE problems are also fixed-parameter tractable with parameter $s + t$ when restricted to $(sP_1 + P_2, K_t - e)$-free graphs. However, the main motivation for our research comes from the area of graph colouring, as we explain in Section 1.3. In Section 3 we give some directions for future research. There, we also show that COLOURING is polynomial-time solvable for $(2P_2, K_t - e)$-free graphs.

Finally, it should be noted that many classes of (H_1, H_2)-free graphs are known to have bounded clique-width (see Section 1.3), in which case it follows that COLOURING is polynomial-time solvable [19]. The graph classes we consider only have bounded clique-width for small values of s and t; the proof of this claim has been omitted due to space restrictions. Thus our results (which are stronger than mere polynomial-time solvability) do not fall into this category.

1.3 Motivation and Related Work

The complexity of MAXIMUM INDEPENDENT SET restricted to H-free graphs has only been partially classified. For instance, the complexity status of this problem on P_5-free graphs is a notorious open case (see [24] for a subexponential algorithm). Because a graph has an independent set of size at least k if and only if its complement has a clique of size at least k, the complexity classification of the CLIQUE problem on H-free graphs is also far from being settled. Our research on the COLOURING problem, which is the main focus of our paper, is well embedded in the literature. As a starting point, Král' et al. [18] completely determined the computational complexity of COLOURING for H-free graphs.

Theorem 1 ([18]). *Let H be a fixed graph. If H is a (not necessarily proper) induced subgraph of P_4 or of $P_1 + P_3$, then* COLOURING *can be solved in polynomial time for H-free graphs; otherwise it is* NP-*complete for H-free graphs.*

Theorem 1 initiated a study of the computational complexity of the k-COLOURING problem on H-free graphs. This classification is still open (see [12] for a survey). Due to Theorem 1, the COLOURING problem restricted to graph classes characterized by *two* forbidden induced subgraphs has received a significant amount of attention as well. We survey known results below.

We observe that Theorem 1 implies that COLOURING is polynomial-time solvable for (H_1, H_2)-free graphs if one of H_1, H_2 is an induced subgraph of P_4 or of $P_1 + P_3$. Another straightforward case is as follows. For positive integers s and t, the *Ramsey number* $R(s,t)$ is the smallest number n such that all graphs on n vertices contain an independent set of size s or a clique of size t. Ramsey's Theorem (see e.g. [10]) states that such a number exists for all positive integers s and t. As an immediate consequence, COLOURING is polynomial-time solvable on (sP_1, K_t)-free graphs for all s and t. Hence, it is natural to consider graph classes that can be obtained by adding one edge to the first forbidden induced subgraph and removing one edge from the second. This leads to the class of $(sP_1 + P_2, K_t - e)$-free graphs; note that this class includes all $(sP_1, K_t - e)$-free graphs and all $(sP_1 + P_2, K_{t-1})$-free graphs. As explained in Section 1.2, we have

fpt algorithms with parameter $s+t$ for solving COLOURING on (sP_1+P_2, K_t-e)-free graphs (and consequently for (sP_1, K_t-e)-free graphs and for (sP_1+P_2, K_t)-free graphs). This result adds to the body of existing work on COLOURING for (H_1, H_2)-free graphs, which we further discuss below.

The following result, which we will use later on, is due to Gyárfás [15].

Theorem 2 ([15]). *Let $\ell, t \geq 1$ be two integers. Then every (P_ℓ, K_t)-free graph can be coloured with at most $(\ell - 1)^{t-2}$ colours.*

Theorem 2 was slightly improved by Gravier, Hoáng and Maffray [14], who showed that every (P_ℓ, K_t)-free graph that is not a complete graph can be coloured with at most $(\ell - 2)^{t-2}$ colours. Each of these two results implies that COLOURING is polynomial-time solvable on (F, K_t)-free graphs, whenever F is the disjoint union of one or more paths such that k-COLOURING is polynomial-time solvable on F-free graphs for all $k \geq 1$. Combining this observation with such existing results for k-COLOURING [6,9,16] gives us a number of polynomial-time solvable cases [11]. Also, the fact that COLOURING can be solved in polynomial time on graphs of bounded clique-width [19] directly leads to polynomial-time results for COLOURING restricted to $(K_{1,3}, C_3 + P_1)$-free graphs [1], $(P_1 + P_4, \overline{P_1 + P_4})$-free graphs [3], $(P_5, \overline{P_1 + P_4})$-free graphs [2] and $(P_1 + P_4, \overline{P_5})$-free graphs [2]. Here, the graph $K_{1,r}$ denotes the graph with vertices $u, v_1, \ldots, v_r$ and edges $uv_1, \ldots, uv_r$. Other results on COLOURING for (H_1, H_2)-free graphs can be found in [4,5,9,17,18,20,21,22,25], all of which are summarized in Theorem 3 together with the above results and a weaker formulation of our new result (Statement (ii)-8). For details we refer to Golovach and Paulusma [11], who formulate the same theorem without Statement (ii)-8. In this theorem, C_3^+ denotes the *paw*, which is the graph with vertices a, b, c, d and edges ab, ac, ad, bc.

Theorem 3. *Let H_1 and H_2 be two fixed graphs. Then the following holds:*

(i) COLOURING *is* NP-*complete for (H_1, H_2)-free graphs if*

    1. $H_1 \supseteq_i C_r$ for some $r \geq 3$ and $H_2 \supseteq_i C_s$ for some $s \geq 3$
    2. $H_1 \supseteq_i K_{1,3}$ and $H_2 \supseteq_i K_{1,3}$
    3. H_1 and H_2 contain a spanning subgraph of $2P_2$ as an induced subgraph
    4. $H_1 \supseteq_i C_3$ and $H_2 \supseteq_i K_{1,r}$ for some $r \geq 5$
    5. $H_1 \supseteq_i C_r$ for $r \geq 4$ and $H_2 \supseteq_i K_{1,3}$
    6. $H_1 \supseteq_i C_3$ and $H_2 \supseteq_i P_{164}$
    7. $H_1 \supseteq_i C_r$ for $r \geq 5$ and H_2 contains a spanning subgraph of $2P_2$ as an induced subgraph
    8. $H_1 \supseteq_i C_r + P_1$ for $3 \leq r \leq 4$ or $H_1 \supseteq_i \overline{C_r}$ for $r \geq 6$, and H_2 contains a spanning subgraph of $2P_2$ as an induced subgraph
    9. $H_1 \supseteq_i K_4$ or $H_1 \supseteq_i K_4 - e$, and $H_2 \supseteq_i K_{1,3}$.

(ii) COLOURING *is polynomial-time solvable for (H_1, H_2)-free graphs if*

    1. H_1 or H_2 is an induced subgraph of $P_1 + P_3$ or of P_4
    2. $H_1 \subseteq_i K_{1,3}$ and $H_2 \subseteq_i C_3 + P_1$
    3. $H_1 \neq K_{1,5}$ is a forest on at most six vertices and $H_2 \subseteq_i C_3^+$

4. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 \subseteq_i C_3^+$
5. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 = K_t$ for $t \geq 4$
6. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_1 + P_4}$
7. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i 2P_2$, and $H_2 \subseteq_i \overline{P_5}$
8. $H_1 \subseteq_i sP_1 + P_2$ for $s \geq 0$ or $H_1 = 2P_2$, and $H_2 \subseteq_i K_t - e$ for $t \geq 2$.

The following result is the only parameterized result known for COLOURING of H-free graphs. Recall that k is the number of colours permitted.

Theorem 4 ([7]). *The* COLOURING *problem on* $(sP_1 + P_2)$*-free graphs is fixed-parameter tractable with parameter* $k + s$.

Very few parameterized results on INDEPENDENT SET are known [8]. In particular, we need the following one. Recall that k is the minimum number of independent vertices required.

Theorem 5 ([8]). *The* INDEPENDENT SET *problem on* $(K_t - e)$*-free graphs is fixed-parameter tractable with parameter* $k + t$.

We remark that the running times of the algorithms of Theorems 4 and 5 are $f(k+s)n^{O(1)}$ and $g(k+t)n^{O(1)}$, respectively, with k in the exponent of both f and g, whereas in our setting k is part of the input.

2 The Proofs of Our Results

In this section, we show that the COLOURING, INDEPENDENT SET and CLIQUE problems on $(sP_1 + P_2, K_t - e)$-free graphs are fixed-parameter tractable with parameter $s + t$. We need the following additional terminology.

Let $G = (V, E)$ be a graph. Then $N(u) = \{v \in V \mid uv \in E\}$ is the *neighbourhood* of $u \in V$. For $S \subseteq V$, we write $N(S) = \{v \in V \setminus S \mid uv \in E$ for some $u \in S\}$. A subset $M \subseteq E$ is a *matching* if no two edges in M share an end-vertex. A matching M is A-*saturating* for some subset $A \subseteq V$ if every vertex of A is an end-vertex of some edge in M; if M is V-saturating, then M is a *perfect* matching. A graph is *p-partite* if its vertex set can be partitioned into at most p independent sets, which we call *partition classes*. If $p = 2$, the graph is *bipartite*. The complement of a p-partite graph is called a *co-p-partite graph* (whose partition classes are cliques).

Before stating the proofs of our results, let us first give an outline. Because a graph is $(sP_1 + P_2, K_t - e)$-free if and only if its complement is $((t-2)P_1 + P_2, K_{s+2} - e)$-free, the results for the CLIQUE problem follow immediately from those for the INDEPENDENT SET problem. In Lemma 1 we show that every $(sP_1 + P_2, K_t - e)$-free graph is $((s+1)P_1, K_t - e)$-free or $(sP_1 + P_2, K_{s^2(t-3)+2})$-free. This enables us to do as follows. We first show our results for COLOURING and INDEPENDENT SET on $(sP_1 + P_2, K_t)$-free graphs in Lemmas 2 and 3, respectively, and on $(sP_1, K_t - e)$-free graphs in Lemmas 8 and 4, respectively. To prove these lemmas, we will use Theorems 2, 4 and 5. We also use Lemma 2 to prove Lemma 3 and Lemma 4, along with some structural results (Lemmas 5–7)

to prove Lemma 8. We then combine our intermediate steps to prove Theorem 6, in which we state our main results.

As noted, we start with Lemma 1.

Lemma 1. *Let G be a (sP_1+P_2, K_t-e)-free graph. Then G is $((s+1)P_1, K_t-e)$-free or $(sP_1 + P_2, K_{s^2(t-3)+2})$-free.*

Proof. Let $G = (V, E)$ be a $(sP_1 + P_2, K_t - e)$-free graph. Suppose that G is not $(s + 1)P_1$-free. We will show that G must then be $K_{s^2(t-3)+2}$-free.

Because G is not $(s+1)P_1$-free, G contains an independent set S on at least $s + 1$ vertices. We assume that S is maximal (with respect to set inclusion). Let $u_1, \ldots, u_p$ be the vertices of S for some $p \geq s + 1$. Let $X_1 = N(u_1)$, and for $i = 2, \ldots, p$, let X_i denote the set of vertices in $V \setminus S$ that are adjacent to u_i but not to any vertex in $\{u_1, \ldots, u_{i-1}\}$. By maximality of S, every vertex of G is either in S or in some X_i.

We claim that $X_i = \emptyset$ for $i \geq s + 1$. Indeed, if $i \geq s + 1$ and $x \in X_i$ then $G[\{u_1, \ldots, u_s, x, u_i\}]$ would be isomorphic to $sP_1 + P_2$.

Now suppose that for some i, X_i contains a clique K on at least $s(t - 3) + 1$ vertices. If $t - 2$ vertices of K were adjacent to u_j for some $j \neq i$, then these $t - 2$ vertices, together with u_i and u_j, would induce a $K_t - e$ in G. Therefore, for each $j \neq i$, at most $t - 3$ vertices of K can be adjacent to u_j. Hence, there must be a vertex $x \in K$ that is not adjacent to any vertex in $\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{s+1}\}$. However, then $G[\{u_1, \ldots, u_{s+1}, x\}]$ is isomorphic to $sP_1 + P_2$. This is a contradiction. Thus all sets $X_1, \ldots, X_s$ can only contain cliques of size at most $s(t - 3)$. Because S is an independent set and $X_i = \emptyset$ for $i \geq s + 1$, this means that the largest clique in G has size at most $s^2(t - 3) + 1$. We conclude that G is $K_{s^2(t-3)+2}$-free, as desired. $\qquad\square$

We are now ready to prove Lemmas 2 and 3.

Lemma 2. *The* COLOURING *problem on $(sP_1 + P_2, K_t)$-free graphs is fixed-parameter tractable with parameter $s + t$.*

Proof. We observe that every $(sP_1 + P_2, K_t)$-free graph is (P_{2s+2}, K_t)-free, and consequently can be coloured with at most $(2s+1)^{t-2}$ colours due to Theorem 2. Hence we can apply Theorem 4. In fact, Theorem 4 gives us an explicit optimal colouring, rather than just the chromatic number. Hence, even the problem of finding an optimal colouring in a $(sP_1 + P_2, K_t)$-free graph is fixed-parameter tractable with parameter $s + t$. $\qquad\square$

Lemma 3. *The* INDEPENDENT SET *problem on $(sP_1 + P_2, K_t)$-free graphs is fixed-parameter tractable with parameter $s + t$.*

Proof. Let G be a $(sP_1 + P_2, K_t)$-free graph on n vertices. By (the proof of) Lemma 2, we can find an optimal colouring c of G in fpt time with parameter $s + t$. Because G is (P_{2s+2}, K_t)-free, c uses $k \leq (2s + 1)^{t-2}$ colours due to Theorem 2. Let $C_1, \ldots, C_k$ be the colour classes of c. We may assume that C_1 is a maximal independent set (with respect to set inclusion) in G, and that for

$i = 2, \ldots, k$, the set C_i is a maximal independent set in $G \setminus (C_1 \cup \cdots \cup C_{i-1})$. Indeed, if some $x \in C_i$ has no neighbours in C_j for some $j < i$, we can move x to C_j in order to obtain another optimal colouring of G.

We branch by choosing an index b to be the largest index such that C_b contains a vertex of the maximum independent set I that we are searching for. There are at most $(2s + 1)^{t-2}$ ways of doing this. We then branch further by choosing a vertex x in C_b that we assume will be in I. This leads to at most n branches altogether. By maximality, x has a neighbour in C_a for every $a < b$. Let J_a be the set of vertices in C_a that are not adjacent to x. Because G is $(sP_1 + P_2)$-free, J_a contains at most $s - 1$ vertices. Let $H_x = J_1 \cup \cdots \cup J_{b-1}$. By the definition of the sets J_i and the choice of x, we find that $I \setminus H_x \subseteq C_b$. We branch further by choosing an independent set $I' \subseteq H_x$. Because H_x has size at most $\beta = (b-1)(s-1) \le ((2s+1)^{t-2} - 1)(s-1)$, there are at most 2^β ways of doing this. We then extend I' by adding all vertices of C_b that do not have a neighbour in I'. The final independent set is a candidate for being a maximum independent set. After considering all independent sets obtained in this way, we choose one that has maximum size. Because we considered all possible ways of constructing a maximum independent set, the above algorithm is correct. Because our algorithm constructs at most $n2^\beta$ independent sets, it runs in fpt time when parameterized by $s + t$. $\qquad\square$

Here is Lemma 4, the proof of which we have omitted.

Lemma 4. *The* INDEPENDENT SET *problem on* $(sP_1, K_t - e)$*-free graphs is fixed-parameter tractable with parameter* $s + t$.

To prove Lemma 8 we need three structural lemmas, the first of which is well-known and can be found in [10].

Lemma 5 (Hall's Marriage Theorem). *A bipartite graph G with vertex partition $A \cup B$ has an A-saturating matching if and only if $|N(X)| \ge |X|$ for all $X \subseteq A$.*

Lemma 6 follows from Lemma 5; we omit the proof details. Note that in a bipartite graph with partition classes A and B an A-saturating matching is perfect if $|A| = |B|$.

Lemma 6. *Let G be a bipartite graph with partition classes A and B. Let p, q, n be integers such that $|A| = |B| = n \ge p + q$. If every vertex in A has degree at least $n - p$ and every vertex in B has degree at least $n - q$, then G contains a perfect matching.*

We need Lemma 6 to prove Lemma 7. Lemma 7 is a key lemma. It gives us a sufficient condition on the number of edges that we may allow between mutually vertex-disjoint cliques without increasing the chromatic number of their union.

Lemma 7. *Let k, a, b be integers such that $k \ge 2a(b-1)$. Let G be a co-b-partite graph with partition classes $X_1, \ldots, X_b$, all of size at most k. If every vertex in X_i has at most a neighbours in X_j for all $1 \le i, j \le b$ when $i \ne j$, then G is k-colourable.*

Proof. Without loss of generality, we may assume that every clique X_i contains exactly k vertices. We use induction on b. The case $b = 1$ is trivial. Let $b \geq 2$. Let $G' = G[X_1 \cup \cdots \cup X_{b-1}]$. Because $k \geq 2a(b-1) \geq 2a(b-2)$, we can apply our induction hypothesis to find that G' is k-colourable. Let c be a k-colouring of G', and let $X_b = \{x_1, \ldots, x_k\}$. We construct an auxiliary bipartite graph F as follows. For each colour $1 \leq i \leq k$ we create a vertex u_i. This yields the set $U = \{u_1, \ldots, u_k\}$. For each vertex $x_j \in X_b$ we introduce a copy x_j'. This yields the set $X = \{x_1', \ldots, x_k'\}$. The partition classes of F are U and X. We add an edge from u_i to x_j' in F if and only if c does not assign colour i to any neighbour of x_j. We observe that c can be extended to a k-colouring of G if and only if F has a perfect matching. Hence, it remains to show that this is indeed the case.

Because every X_i is a clique of size k, every colour of c occurs $b - 1$ times. Recall that we assume that every vertex in X_i has at most a neighbours in X_j for all $1 \leq i, j \leq b$, where $i \neq j$. By combining these two facts we find that every u_i has degree at least $k - a(b-1)$ and that every x_j' has degree at least $k - a(b-1)$. As $k \geq 2a(b-1) = a(b-1) + a(b-1)$, we may apply Lemma 6 to find that F has a perfect matching. $\qquad\square$

We are now ready to prove Lemma 8.

Lemma 8. *The* Colouring *problem on* $(sP_1, K_t - e)$*-free graphs is fixed-parameter tractable with parameter* $s + t$.

Proof. Let G be an $(sP_1, K_t - e)$-free graph on n vertices. We may assume without loss of generality that $s \geq 2$ and $t \geq 3$, as the proof is straightforward for $s \leq 1$ or $t \leq 2$. We first find a maximum independent set S of G. According to (the proof of) Lemma 4, we can do this in fpt time with parameter $s + t$.

We may assume without loss of generality that S is of size $s - 1$; otherwise G is $((s-1)P_1, K_t - e)$-free. Let $u_1, \ldots, u_{s-1}$ be the vertices of S. For $1 \leq i < j \leq s - 1$, let $X_{i,j}$ be the set of vertices adjacent to both u_i and u_j. If some $X_{i,j}$ contains a clique on $t - 2$ vertices, then the vertices of this clique together with u_i, u_j form an induced $K_t - e$. Hence, every $X_{i,j}$ is (sP_1, K_{t-2})-free. Recall that $R(s,t)$ is the Ramsey number for integers s and t. Because $X_{i,j}$ is (sP_1, K_{t-2})-free, $X_{i,j}$ contains at most $R(s, t-2) - 1$ vertices. Let $D = \bigcup X_{i,j}$. Then $|D| \leq \binom{s-1}{2}(R(s, t-2) - 1)$. Hence, the size of D is bounded by a function of s and t.

Let X_i consist of u_i and those vertices that are adjacent to u_i but not to any other vertex in S. Each X_i must be a clique, since if $x, y \in X_i$ were non-adjacent, then $\{x, y\} \cup S \setminus \{u_i\}$ would be an independent set larger than S, contradicting the fact that S is a maximum independent set. Note that every vertex of G is either in D or in some X_i (see Fig. 1). Hence, we find that the vertices of G can be partitioned into $s - 1$ cliques $X_1, \ldots, X_{s-1}$ and a set D. However, from these s sets, we only know that D has bounded size (in terms of s and t).

Now suppose that some vertex $x \in X_i$ had $t - 2$ neighbours in X_j for some $j \neq i$. Then these $t - 2$ neighbours, together with x and u_j, would induce a $K_t - e$ in G. Hence, every vertex in X_i has at most $t - 3$ neighbours in every
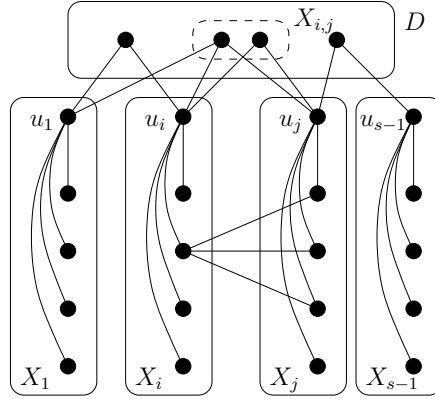
Fig. 1. Decomposition of the graph G into sets $X_1, \ldots, X_{s-1}, D$.

X_j with $j \neq i$. Consequently, each vertex in X_i has at most $(t-3)(s-1) + |D|$ neighbours outside of X_i.

We now start a branching procedure by first colouring the vertices of D in every possible way using colours from the set $C = \{1, \ldots, |D|\}$. After colouring D, for each $i = 1, \ldots, s - 1$, we choose a subset $C_i \subseteq C$ of size $|C_i| \leq |X_i|$, which we let consist of exactly those colours from C that will occur on X_i. We branch over all possibilities for choosing such sets C_i. After choosing the sets C_i we branch further. For all cliques X_i of size at most $\gamma = (|D| + 2)(s - 2)(t - 3) + (t - 3)|D| + |D|$ (we explain this number later) we branch by trying every possible way of colouring a subset of X_i of size $|C_i|$ with the colours from C_i. This yields a partial colouring of G. The total number of these partial colourings is at most

$$|D|^{|D|} \prod_{i=1}^{s-1} 2^{|D|} \binom{\gamma}{|C_i|} |C_i|^{|C_i|},$$

which only depends on s and t, and which may be strictly less because whenever two adjacent vertices are assigned the same colour, we naturally cut the branch.

Now let c be a partial colouring that is obtained in this way. Assume that c uses k_c colours. We let $\texttt{total}(c)$ denote the smallest number of colours of a colouring c' of G subject to the following two conditions:

(i) c' extends c;
(ii) c' does not use any colours from $C \backslash C_i$ on vertices of X_i.

Let Z be the set of vertices that are not coloured by c. We may assume without loss of generality that $\beta = |X_1| - |C_1| = \max\{|X_i| - |C_i| \mid 1 \leq i \leq s - 1\}$.

First suppose that X_1 has size at most γ, in which case Z has size at most $\sum_i |(X_i| - |C_i|) \leq (s - 1)\gamma$. By definition of the sets C_i, we are required to use colours on Z that are not in C. In other words, we are to colour Z independently from the way we coloured the rest of G. Because in this case $|Z|$ only depends

on s and t, we use brute force to compute the chromatic number $\chi_{G[Z]}$ of $G[Z]$. Hence we find that $\texttt{total}(c) = k_c + \chi_{G[Z]}$ in this case.

Now suppose that X_1 has size at least $\gamma + 1$. We observe the following. Let v be any vertex in D. If v is adjacent to all vertices of a clique X_i, then $c(v)$ does not appear in C_i, as otherwise we can cut the branch. If v is adjacent to a set X_i' of $t - 2$ vertices of a clique X_i but not to some vertex $w \in X_i$, then $X_i' \cup \{v, w\}$ induce a $K_t - e$ in G, which is not possible. Hence, we may assume without of generality that every vertex in D is adjacent to at most $t - 3$ vertices in every clique X_i. This means that the total number of vertices in X_i that have a neighbour in D is at most $(t - 3)|D|$. Also recall that every vertex in X_i has at most $t - 3$ neighbours in any X_j with $j \neq i$. We may assume without loss of generality that for some $h \leq s - 1$, the cliques $X_1, \ldots, X_h$ are precisely those for which $|X_i| \geq \gamma + 1$. We apply the following greedy approach for assigning colours from C_i to every clique X_i with $1 \leq i \leq h$. We arbitrarily colour a set X_1^* of $|C_1|$ vertices of X_1 that are not adjacent to any vertices in D with colours from C_1. We then arbitrarily colour a set X_2^* of $|C_2|$ vertices of X_2 that are not adjacent to any vertices in $D \cup X_1^*$ with colours from C_2, and so on, until we have processed X_h. We can follow this greedy approach, because a clique X_i with $1 \leq i \leq h$ has a size that is sufficiently large, that is, at least $\gamma + 1 = (|D| + 2)(s - 2)(t - 3) + (t - 3)|D| + |D| + 1$. (Note that we could have chosen γ to be smaller here, but this value simplifies the arguments in the next paragraph.) Let c^* denote the resulting partial colouring. Note that c^* extends c.

Assume that c^* uses k_{c^*} colours. We claim that $\texttt{total}(c) = k_{c^*} + |X_1| - |C_1|$. Note that $\texttt{total}(c) < k_{c^*} + |X_1| - |C_1|$ is not possible, because of condition (ii) and the fact that X_1 is a clique. Hence, we are left to show that all uncoloured vertices of G can be coloured with at most $|X_1| - |C_1|$ colours. This follows immediately from Lemma 7 by taking $k = |X_1| - |C_1|$, $a = t - 3$ and $b = s - 1$. We may apply this lemma for the following two reasons. First, for $i = 1, \ldots, h$, we have $|X_i| - |C_i| \leq |X_1| - |C_1|$ by definition. Second, we have $|X_1| - |C_1| \geq \gamma + 1 - |C_1| \geq 2(s - 2)(t - 3)$.

As we branched in all possible ways, we find that the smallest $\texttt{total}(c)$ is the chromatic number of G. Note that our algorithm runs in fpt time with parameter $s + t$, as required, and that it also produces an optimal colouring of G. $\qquad \square$

We are now ready to state and prove our main theorem.

Theorem 6. *The* Colouring, Independent Set *and* Clique *problems on* $(sP_1 + P_2, K_t - e)$-*free graphs are fixed-parameter tractable with parameter* $s + t$.

Proof. First recall the following. Because a graph is $(sP_1 + P_2, K_t - e)$-free if and only if its complement is $(P_2 + (t - 2)P_1, K_{s+2} - e)$-free, we only have to consider the Independent Set and Colouring problems.

Let G be a $(sP_1 + P_2, K_t - e)$-free graph. We start by checking whether G contains an independent set on $s + 1$ vertices. We can do this in fpt time with parameter $s + t$ by Theorem 5. If it does, then G is not $(s + 1)P_1$-free. By Lemma 1, this means that G must be $(sP_1 + P_2, K_{s^2(t-3)+2})$-free. In this case, we

can solve Colouring and Independent Set by Lemmas 2 and 3, respectively. Otherwise, that is, if G contains no independent set on $s + 1$ vertices, then G is $((s + 1)P_1, K_t - e)$-free. In that case, we can solve Independent Set and Colouring by Lemmas 4 and 8, respectively. $\qquad\square$

3   Final Remarks

The ultimate goal is to complete Theorem 3. This requires new proof techniques to deal with a number of nontrivial cases, such as when H_1 is the claw $K_{1,3}$ and H_2 is a long path. As regards our result it seems natural to settle, as a next step, the complexity status of Colouring for $(sP_2, K_t - e)$-free graphs. Our next result (proof omitted) shows that the case $s = 2$ is polynomial-time solvable.

Theorem 7. *The* Colouring *problem on* $(2P_2, K_t - e)$-*free graphs can be solved in polynomial time for all* $t \geq 1$.

Another possible generalization of our result on Colouring is to consider the following variant of graph colouring. In Precolouring Extension we assume that a (possibly empty) subset $W \subseteq V$ of a graph $G = (V, E)$ is precoloured by a *precolouring* $c_W : W \to \{1, 2, \ldots, k\}$ for some given integer k, and the question is whether we can extend c_W to a k-colouring of G. The classification of Precolouring Extension on H-free graphs is known [13]. However, the classification of Precolouring Extension on (H_1, H_2)-free graphs is still open. In this respect, we note that our results cannot be generalized to another even more general variant of graph colouring called list colouring. A *list assignment* of a graph $G = (V, E)$ is a function $\mathcal{L}$ that assigns a list $L(u)$ of so-called *admissible* colours to each $u \in V$. We say that a colouring $c : V \to \{1, 2, \ldots\}$ *respects* $\mathcal{L}$ if $c(u) \in L(u)$ for all $u \in V$. The List Colouring problem is that of testing whether a given graph has a colouring that respects some given list assignment. Golovach and Paulusma [11] completely classified the complexity of the List Colouring problem for (H_1, H_2)-free graphs by showing that this problem is polynomial-time solvable for (H_1, H_2)-free graphs in the following three cases: (i) $H_1 \subseteq_i P_3$ or $H_2 \subseteq_i P_3$, (ii) $H_1 \subseteq_i C_3$ and $H_2 \subseteq_i K_{1,3}$ and (iii) $H_1 = sP_1$ for $s \geq 3$ and $H_2 = K_t$ for $t \geq 3$, whereas it is NP-complete for all other pairs (H_1, H_2).

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