Parameterized Algorithms for the Independent Set Problem in Some Hereditary Graph Classes

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Abstract. The maximum independent set problem is NP-complete for graphs in general, but becomes solvable in polynomial time when restricted to graphs in many special classes. The problem is also intractable from a parameterized point of view. However, very little is known about parameterized complexity of the problem in restricted graph classes. In the present paper, we analyse two techniques that have previously been used to solve the problem in polynomial time for graphs in particular classes and apply these techniques to develop fpt-algorithms for graphs in some classes where the problem remains NP-complete.

Keywords: Independent set; Fixed-parameter tractability; Augmenting graph; Modular decomposition; Hereditary class of graphs

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1 Introduction

We study simple undirected graphs without loops or multiple edges. In a graph, an independent set is a subset of vertices, no two of which are adjacent and a clique is a subset of pairwise adjacent vertices. The size of a maximum independent set in a graph G is called the *independence number* of G and is denoted $\alpha(G)$, while the size of a maximum clique is called the *clique number* of G and is denoted $\omega(G)$.

The MAXIMUM INDEPENDENT SET problem is that of finding an independent set of maximum size in a graph. From a computational point of view, this is a difficult problem, i.e. it is NP-hard. Moreover, it remains NP-hard under substantial restrictions, for instance, for triangle-free graphs [27] and for graphs of vertex degree bounded by d, where $d \geq 3$. On the other hand, in many special graph classes the problem admits polynomial-time algorithms, which is the case for perfect graphs [16], claw-free graphs [24], and graphs of bounded clique-width [7].

In this paper, we study the following parameterization of the MAXIMUM INDEPENDENT SET problem:

k-INDEPENDENT SET
Instance: A graph G and a positive integer k.
Parameter: k.
Problem: Decide whether G has an independent set of size k and find such a set if it exists.

An approach to deal with NP-complete problems in practice is to split a problem that contains a parameter as part of the input into sub-problems for each value of this parameter. A parameterized problem is said to be *fixed-parameter tractable (fpt)* if it can be solved in time $f(k)p(n)$ on instances of input size n, where $f(k)$ is an efficiently computable function, depending only on the value of the parameter k and $p(n)$ is a polynomial independent of k.

Unfortunately, the MAXIMUM INDEPENDENT SET problem remains difficult even under this relaxation. More formally, it is W[1]-hard [10]. However, for graphs in some restricted classes the problem becomes fixed-parameter tractable. In particular, this is the case for graphs without large cliques, which follows from a Ramsey argument (see e.g. [29]). This argument alone implies fixed-parameter tractability of the problem for graphs of bounded degree, of bounded degeneracy, of bounded chromatic number, in all proper minor-closed graph classes (which includes, in particular, classes of graphs excluding single-crossing graphs as minors [8]), all proper classes closed under taking subgraphs (not necessarily induced). Beyond this argument, very little is known on the parameterized complexity of the problem in restricted graph families. Other classes where the problem is known to be fixed-parameter tractable are the complements of t-multiple-interval graphs [13], segment intersection graphs with a bounded number of directions [19] and graphs whose vertices can be partitioned into two subsets of which one induces a graph of bounded clique number and the other induces a graph of bounded independence number [20].

In search of new fpt results, we analyse algorithmic techniques which are traditionally used for obtaining polynomial-time solutions for the MAXIMUM INDEPENDENT SET problem on graphs in special classes. In particular, we study the augmenting graph approach and the modular decomposition technique and apply them to develop fpt-algorithms that solve the problem in several new classes of graphs, generalising some of the previously known results.

All classes considered in this paper are hereditary, in the sense that for any graph G in such a class, all induced subgraphs of G are also in the class. It is known that a class of graphs is hereditary if and only if it can be characterised by a set of forbidden induced subgraphs. We denote the set of graphs containing no induced subgraphs from a set M by $Free(M)$ and call graphs in this class M-free.

For a graph G we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$ respectively. If v is a vertex of G, then $N(v)$ is the neighbourhood of v (i.e. the set of vertices adjacent to v) and $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v. For a subset $U \subseteq V(G)$ we let $G[U]$ be the subgraph of G induced by U, and $N(U)$ be the neighbourhood of U, i.e. the set of vertices outside U that have at least one neighbour in U. By $R(r, s)$ we denote the Ramsey number, i.e. the minimum number n such that every graph with at least n vertices has either an independent set of size r or a clique of size s. As usual, K_n, C_n and P_n denote the complete graph, the chordless cycle and the chordless path on n vertices, respectively. We denote the graph obtained from K_n by deleting an edge by $K_n - e$.

2 The Augmenting Graph Technique

The idea of augmenting graphs was proposed by Berge [3] and then implemented by Edmonds [11] to solve the maximum matching problem, which is equivalent to the MAXIMUM INDEPENDENT SET problem restricted to the class of line graphs. With this restriction, the idea reduces to finding augmenting chains. However, the notion of an augmenting graph lies in the basis of a general approach to solve the problem, which can be described as follows:

Let G be a graph and S an independent set in G. We shall call the vertices of S *white* and the remaining vertices of G *black*.

Definition 1. *An* augmenting graph for S in G *is an induced bipartite subgraph* $H = (W, B, E)$ *of* G*, where* $W \cup B$ *is the bipartition of its vertex set and* E *its edge set, such that:*

- $|B| > |W|$,
- $W \subseteq S$,
- $B \subseteq V(G) \backslash S$, *and*
- $N(B) \cap S \subseteq W$.

Clearly if $H = (W, B, E)$ is an augmenting graph for S, then S is not a maximum independent set in G, since the set $S' = (S \setminus W) \cup B$ is independent and $|S'| > |S|$. Conversely, if S is not a maximum independent set, and S' is a larger independent set, then the subgraph of G induced by the set $(S \setminus S') \cup (S' \setminus S)$ is augmenting for S. Thus we have the following theorem:

Theorem of Augmenting Graphs. *An independent set* S *in a graph* G *is maximum if and only if there are no augmenting graphs for* S *in* G*.*

This theorem suggests the following general approach to find a maximum independent set in a graph G: Begin with any independent set S in G and, as long as S admits an augmenting graph H, augment S as above. This approach has proven to be a useful tool to develop approximate solutions to the problem [18], to compute bounds on the independence number [9], and to solve the problem in polynomial time for graphs in special classes such as claw-free graphs [24], fork-free graphs [1] and some others [2,4,22,26]. In the present paper, we use the idea of augmenting graphs to derive the following fpt result:

Theorem 1. *The* k-INDEPENDENT SET *problem can be solved for* $(K_r - e)$*-free* n*-vertex graphs in time* $f(k, r)p(n)$*, where* $f(k, r)$ *is a function of* k *and* r *only and* $p(n)$ *is a polynomial independent of* k *and* r*.*

Proof. Let G be a $(K_r - e)$-free graph with n vertices and let S be an independent set in G. We assume that S is maximal with respect to set-inclusion and admits no augmenting P_3. Obviously such a set can be found in polynomial time. If $|S| \geq k$, we are done. Therefore, we suppose that $|S| < k$ and explain how to determine whether G admits an augmenting graph in time $g(k, r)p(n)$, which clearly implies the desired result for $f(k, r) = kg(k, r)$. We split the process of finding an augmenting graph into the following general steps.

Step 1: Partition the set of black vertices of G into subsets called *node classes* putting two vertices in the same node class if and only if they have the same neighbourhood in S. Note that there are at most $2^{|S|} - 1 < 2^k$ node classes. We call a node class *light* if its neighbourhood in S contains exactly 1 vertex and *heavy* otherwise. Since S admits no augmenting P_3, every light node class is a clique. Clearly, this step can be implemented in polynomial time.

Step 2: Consider a heavy node class C. Since G is $(K_r - e)$-free, the subgraph of G induced by C must be K_{r-2}-free. Therefore, if $|C| \geq R(k, r-2)$, then C necessarily has an independent set of size k. Thus, in this case we can arbitrarily choose $R(k, r-2)$ vertices in the class and then find an independent set of size k among them in time bounded by a function of k and r. Otherwise, the size of every heavy node class is less than $R(k, r-2)$, in which case the total number of vertices in heavy node classes is bounded by a function of k and r.

Step 3: Generate all independent sets contained in the union of the heavy node classes. From the previous step it follows that the number of such sets and the time needed to generate all of them is bounded by a function of k and r. For each independent set T found in this step, execute Step 3.1.

Step 3.1: If the size of T is strictly larger than the number of its white neighbours, then $G[T \cup (N(T) \cap S)]$ is an augmenting graph. Otherwise, extend T by adding to it some vertices from the light node classes. To this end, delete from the light node classes those vertices that have neighbours in T and then split the thus-modified light classes into two groups: those containing at most kr vertices, we call them *small*, and those containing more than kr vertices, called *large*. Let s be the number of small classes and l the number of large classes. Obviously, $s + l < k$.

Extend T to a larger independent set by adding to it some vertices from small node classes. Since the number of small node classes is at most k and each of them contains at most kr vertices, the number of such extensions and the time needed to find all of them is bounded by a function of k and r. For each such an extension T', execute Step 3.1.1.

Step 3.1.1: If the size of T' is strictly larger than the number of its white neighbours, then $G[T' \cup (N(T') \cap S)]$ is an augmenting graph. Otherwise, extend T' by adding to it vertices from the large node classes. To this end, delete from the large node classes those vertices that have neighbours in T'. Since every light class (small or large) is a clique,

- T' contains at most one vertex in each light class,
- no vertex u from a light node class has more than $r-3$ neighbours in another light node class, since otherwise a $K_r - e$ arises using u, $r-2$ neighbours of u in another light node class and their only neighbour in S.

Therefore, deleting from the large node classes those vertices that have neighbours in T' leaves at least lr vertices in each large node class. Consequently, the set of vertices left in the large node classes necessarily contains an independent set L of size l. This set can be constructed iteratively by picking an arbitrary vertex, deleting its neighbours, and so on. Now we add L to T' and check if $G[(T' \cup L) \cup (N(T' \cup L) \cap S)]$ is an augmenting graph. Observe that it does not matter how we choose L, since for any choice of this set, its neighbourhood in S coincides with the neighbourhood of the large node classes in S.

Summarising, we conclude that determining whether S admits an augmenting graph can be done in time $g(k, r)p(n)$. The number of augmentations to solve the problem is at most k. Therefore, the result follows. □

3 Modular Decomposition

The idea of modular decomposition was first introduced in the 1960s by Gallai [15], and also appeared in the literature under various other names, such as *prime tree decomposition* [12], *X-join decomposition* [17], and *substitution decomposition* [25]. To describe this idea, let us fix some terminology.

Given a graph $G = (V, E)$, a subset of vertices $U \subseteq V$ and a vertex $x \in V$ outside U, we say that x *distinguishes* U if x has both a neighbour and a non-neighbour in U. A subset $U \subseteq V$ is called a *module* of G if no vertex in $V \setminus U$ distinguishes U. A module U is *nontrivial* if $1 < |U| < |V|$, otherwise it is *trivial*. A graph is called *prime* if it has only trivial modules.

An important property of maximal modules is that if G and the complement of G are both connected, then the maximal modules of G are pairwise disjoint. Moreover, from the above definition it follows that if U and W are maximal modules, then either all possible edges between them are present, or none of them are. This property is useful when we deal with the weighted version of the maximum independent set problem.

We say that G is a weighted graph if each vertex v of G is assigned a positive integer $w(v)$, the weight of the vertex. The MAXIMUM WEIGHT INDEPENDENT SET problem is that of finding an independent set of maximum total weight in a weighted graph. This maximum total weight is denoted $\alpha_w(G)$. By using the properties of maximal modules we can find a maximum weight independent set in G by:

(1) recursively solving the problem in the subgraphs of G induced by maximal modules,
(2) contracting each maximal module M to a single vertex and assigning to it the weight $\alpha_w(G[M])$, obtaining in this way a new graph G^0,
(3) solving the problem for the graph G^0.

The graph G^0 constructed in step 2 of the outlined procedure is prime. So the procedure reduces the MAXIMUM WEIGHT INDEPENDENT SET problem from any hereditary class X to prime graphs in X. This reduction can be implemented in polynomial time (see e.g. [23]). In this section we show that this is also an fpt-reduction, i.e. it preserves fixed-parameter tractability. In case of weighted graphs we parameterize the problem by the weight W of a solution. Without loss of generality we will assume that if the input graph has no independent set of weight at least W, the problem asks for an independent set of maximum weight. This generalisation increases the complexity of any algorithm solving the problem at most W times and therefore preserves fixed-parameter tractability. More formally, we consider the following parameterization of the MAXIMUM WEIGHT INDEPENDENT SET problem:

w-INDEPENDENT SET
Instance: A weighted graph G with weight function $w : V(G) \to \{1, 2, 3, \dots\}$ and a positive integer W.
Parameter: W.
Problem: Decide whether G has an independent set of weight at least W and find such a set if it exists. If no such set exists, find an independent set of weight $\alpha_w(G)$ instead.

Theorem 2. *Let $\mathcal{X}$ be a hereditary class of graphs and let $\mathcal{X}_0$ denote the class of prime graphs in $\mathcal{X}$. If the w-INDEPENDENT SET problem is fixed-parameter tractable in $\mathcal{X}_0$, then it is fixed-parameter tractable in $\mathcal{X}$.*

Proof. Let (G, W) be an instance of the w-INDEPENDENT SET problem with $G \in \mathcal{X}$. Let n denote the number of vertices of G. Recall that the modular decomposition tree T of G can be determined in linear time [6, 23] and that the set of leaves of T equals the vertex set V of G. To each node v of T we associate the subgraph G_v of G induced by the leaves of the subtree of T rooted at v. Processing the vertices of T in an order of non-increasing height, for each node v of T we will find an independent set I_v of G_v such that $w(I_v) \geq \min\{W, \alpha_w(G_v)\}$. If the weight of I_v is at least W, we stop the procedure and output I_v. Otherwise, to each node v we assign an independent set I_v of weight $\alpha_w(G_v)$. The procedure starts by assigning to each leaf v of T the independent set $I_v = \{v\}$. Now let v be an inner node of T.

If G_v is disconnected, then the children $v_1, v_2, \ldots, v_l$ of v correspond to the connected components of G_v. In this case, we let $I_v = I_{v_1} \cup I_{v_2} \cup \ldots I_{v_l}$.

If the complement of G_v is disconnected, then the children $v_1, v_2, \ldots, v_l$ of v correspond to the connected components of the complement of G_v. In this case, we let $I_v = I_{v_i}$ where $w(I_{v_i}) = \max\{w(I_{v_1}), w(I_{v_2}), \ldots, w(I_{v_l})\}$.

Finally, if both G_v and its complement are connected, then the children $v_1, \ldots, v_l$ of v correspond to the subgraphs of G_v induced by the maximal modules $U_1, U_2, \ldots, U_l$ of G_v, which partition the vertex set of G_v. Let the graph G_v^0 arise from G_v by contracting each maximal module U_i of G_v into a single vertex, denoted i, to which we assign the weight $w(i) = w(I_{v_i})$. Since G_v^0 belongs to $\mathcal{X}_0$, there is an algorithm $\mathcal{A}$ that solves w-INDEPENDENT SET on the instance (G_v^0, W) in time $f(W)l^c \leq f(W)n^c$, where c is a constant. If I is the output of $\mathcal{A}$, then let $I_v = \bigcup_{i \in I} I_{v_i}$. It is not difficult to see that the set assigned to the root of T correctly solves w-INDEPENDENT SET on the instance (G, W). Since T has $O(n)$ vertices, the overall time complexity is at most $f(W)n^{c+1}$. □

Theorem 2 reduces the w-INDEPENDENT SET problem from general graphs to prime graphs. The corresponding result for the non-parameterized problem is well-known. Our next result shows that the problem can be further reduced to prime graphs containing a clique K_r for a certain value of r.

Theorem 3. *For any $r \in \mathbb{N}$, the w-INDEPENDENT SET problem is fixed-parameter tractable in the class of K_r-free graphs.*

Proof. Let (G, W) be an instance of the w-INDEPENDENT SET problem with $G = (V, E)$ being a K_r-free graph on n vertices. Since the weight of each vertex is a positive integer, the weight of every independent set is at least its size. Therefore, if G has at least $R(r, W)$ vertices it necessarily has an independent set of size (and therefore of weight) at least W. If the number of vertices of G is strictly more than $R(r, W)$, we can delete, without loss of generality, any $n - R(r, W)$ vertices of G, since the remaining vertices of the graph still necessarily have an independent set of size (of weight at least) W.

When the number of vertices of G is bounded by $R(r, W)$, the problem can be solved in time independent of n. This completes the proof. □

To illustrate Theorems 2 and 3, we apply them to solve the w-INDEPENDENT SET problem in the class of (*house*, *bull*)-free graphs. The *house* and *bull* graphs are shown in Fig. 1. Observe that this class contains (C_3, C_4)-free graphs, where the MAXIMUM INDEPENDENT SET problem is NP-hard [27].

Theorem 4. *The w-INDEPENDENT SET problem is fixed-parameter tractable in the class of (house,bull)-free graphs.*

Fig. 1. The house and the bull graphs

Proof: Our proof is based on a characterisation of $(house, bull)$-free graphs from [28]: every prime $(house, bull)$-free graph is either triangle-free or the complement of a bipartite chain graph. (A bipartite graph is a bipartite chain graph if the vertices in both parts of the bipartition are linearly ordered by inclusion of neighbourhoods.) Obviously, for the complements of bipartite graphs, the MAXIMUM WEIGHT INDEPENDENT SET problem can be solved in polynomial time, since the size of any independent set in such a graph is at most 2. Also, by Theorem 3, the w-INDEPENDENT SET problem is fixed-parameter tractable in the class of triangle-free graphs. Therefore, by Theorem 2, it is fixed-parameter tractable in the class of $(house, bull)$-free graphs. □

4 Concluding Remarks and Open Problems

In this paper, we used the augmenting graph technique and modular decomposition to obtain new results on the parameterized complexity of the MAXIMUM INDEPENDENT SET problem in hereditary classes of graphs. The new results together with some previously known results allow us to conclude, in particular, that the problem is fixed-parameter tractable in all hereditary classes defined by a single forbidden induced subgraph G with at most 4 vertices, except for $G = C_4$. Finding the parameterized complexity of the problem in the class of C_4-free graphs is a challenging open problem. In addition to the two techniques studied in this paper, some other approaches can be useful for finding an answer to the above question, such as graph transformations [21], separating cliques [5] and split decomposition [30].

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