

On factorial properties of chordal bipartite graphs

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Abstract

For a graph property X , let X_n be the number of graphs with vertex set $\{1, \dots, n\}$ having property X , also known as the speed of X . A property X is called factorial if X is hereditary (i.e. closed under taking induced subgraphs) and $n^{c_1 n} \leq X_n \leq n^{c_2 n}$ for some positive constants c_1 and c_2 . Hereditary properties with the speed slower than factorial are surprisingly well structured. The situation with factorial properties is more complicated and less explored. To better understand the structure of factorial properties we look for minimal superfactorial ones. In [16], Spinrad showed that the number of n -vertex chordal bipartite graphs is $2^{\Theta(n \log^2 n)}$, which means this class is superfactorial. On the other hand, all subclasses of chordal bipartite graphs that have been studied in the literature, such as forest, bipartite permutation, bipartite distance-hereditary or convex graphs, are factorial. In the present paper, we study more hereditary subclasses of chordal bipartite graphs and reveal both factorial and superfactorial members in this family. The latter fact shows that the class of chordal bipartite graphs is not a minimal superfactorial one. Finding minimal superfactorial classes in this family remains a challenging open question.

Keywords: Hereditary class of graphs; Speed of hereditary properties; Factorial class; Chordal bipartite graphs

1 Introduction

A *graph property* is an infinite class of graphs closed under isomorphism. Given a property X , we write X_n for the number of graphs in X with vertex set $\{1, 2, \dots, n\}$. Following [4], we call X_n the *speed* of the property X .

A property is *hereditary* if it is closed under taking induced subgraphs. Scheinerman and Zito showed in [15] that for a hereditary property X the growth of X_n is far from arbitrary. In particular, the rates of the growth constitute discrete layers. In [15], the authors distinguish five such layers: constant, polynomial, exponential, factorial and superfactorial. Independently, similar results have been obtained by Alekseev in [2]. Moreover, the latter paper reveals all minimal classes in the first four layers and provides the first three layers with complete structural characterizations. The minimal layer for which no such

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characterization is known is the factorial one. A graph property X is said to be *factorial* if the speed X_n satisfies the inequalities $n^{c_1 n} \leq X_n \leq n^{c_2 n}$ for some positive constants c_1 and c_2 .

The factorial layer contains many classes of theoretical or practical importance, such as line graphs, interval graphs, permutation graphs, threshold graphs, forests, planar graphs and, even more generally, all proper minor-closed graph classes [14], all classes of graphs of bounded vertex degree, of bounded clique-width [3], etc. On the other hand, except the definition, very little can be said about the factorial layer in general. There is no membership test or common structural characterization for classes in this layer. To simplify the study of this layer, in [11] the following conjecture was proposed.

Conjecture on factorial properties. *A graph property X is factorial if and only if the fastest of the following three properties is factorial: bipartite graphs in X , co-bipartite graphs in X , split graphs in X .*

We recall that a graph is bipartite if its vertices can be partitioned into at most two independent sets. By a co-bipartite graph we mean the complement of a bipartite graph. Finally, a split graph is a graph whose vertices can be partitioned into an independent set and a clique.

To justify the above conjecture we observe that if in the text of the conjecture we replace the word “factorial” by any of the lower layers (constant, polynomial or exponential), then the text becomes a valid statement. Also, the “only if” part of the conjecture is true, because all minimal factorial classes are subclasses of bipartite, co-bipartite or split graphs. There are 9 such classes of which three are subclasses of bipartite graphs, three are subclasses of co-bipartite graphs and three are subclasses of split graphs. The three minimal factorial classes of bipartite graphs are:

- P^1 , the class of graphs of vertex degree at most 1,
- P^2 , the class of “bipartite complements” of graphs in P^1 , i.e. the class of bipartite graphs in which every vertex has at most one non-neighbor in the opposite part,
- P^3 , the class of $2K_2$ -free bipartite graphs, also known as chain graphs for the property that the neighborhoods of vertices in each part form a chain.

A graph property X is *superfactorial* if for every positive constants c and n_0 there is $n \geq n_0$ such that $X_n \geq n^{cn}$. If we knew all minimal superfactorial classes, proving or disproving the above conjecture would be an easy task. However, none of such classes is known. Under these circumstances, we look for “smallest” classes which are known to be superfactorial. One of such classes is the class of chordal bipartite graphs.

A bipartite graph is *chordal bipartite* if it does not contain chordless cycles of length more than 4. The class of chordal bipartite graphs contains several important subclasses, such as forests, chain graphs, bipartite permutation graphs [17], bipartite distance-hereditary graphs [5], biconvex [1] and convex graphs. All these subclasses are known to be factorial. On the other hand, as shown by Spinrad in [16], the speed of the class of chordal bipartite graphs is $2^{\Theta(n \log^2 n)}$, which means that it is superfactorial. In the attempt to determine whether it is a minimal superfactorial class, in the present paper we study hereditary subclasses of chordal bipartite graphs. Every such a subclass can be obtained by excluding from this class a set of chordal bipartite graphs, i.e. by forbidding

a set of chordal bipartite graphs as induced subgraphs. One of our conclusions is that the class of chordal bipartite graphs is not a minimal superfactorial class. We reveal a proper superfactorial subclass of chordal bipartite graphs in Section 3. On the other hand, in Sections 4 and 5 we identify a number of new factorial members in the family of hereditary subclasses of chordal bipartite graphs.

All preliminary information related to the topic of the paper, including notations and an overview of previously known results, can be found in Section 2. In Section 6 we conclude the paper with a list of open problems.

2 Preliminaries

All graphs in this paper are finite, undirected, without loops or multiple edges. For a graph G we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G respectively. As usual, K_n is a complete graph on n vertices, $K_{n,m}$ is a complete bipartite graph with parts of size n and m , and C_n a chordless cycle of length n . Given two graphs G and H , we denote by $G+H$ the disjoint union of G and H . In particular, nG is the disjoint union of n copies of G .

The subgraph of G induced by a set of vertices $U \subseteq V(G)$ is the graph with vertex set U and two vertices being adjacent if and only if they are adjacent in G . The subgraph of G induced by U will be denoted $G[U]$. If a graph H is isomorphic to an induced subgraph of G , we say that G contains H as an induced subgraph. Otherwise, we say that G is H -free. For a set M , the class of all M -free graphs is denoted $Free(M)$ and we call M the set of forbidden induced subgraphs for this class. It is well-known that a class X of graphs is hereditary if and only if $X = Free(M)$ for some set M .

Every set X of graphs, hereditary or not, can be “approximated” by two hereditary classes as follows: by $\lfloor X \rfloor$ we denote the maximal hereditary subclass contained in X and by $\lceil X \rceil$ the minimal hereditary class containing X . It is not difficult to see that $\lfloor X \rfloor$ and $\lceil X \rceil$ are uniquely defined and $\lfloor X \rfloor \subseteq X \subseteq \lceil X \rceil$ with equalities holding if and only if X is hereditary.

In this paper, the class of our interest is the class of chordal bipartite graphs, which is precisely the class $Free(C_3, C_5, C_6, C_7, \dots)$. Spinrad has shown in [16] that the number of n -vertex graphs in this class is proportional to $2^{\Theta(n \log^2 n)}$, which means that this class is superfactorial. As we mentioned in the introduction, this class contains several interesting and important subclasses, such as forests, chain graphs, bipartite permutation graphs [17], bipartite distance-hereditary graphs [5], biconvex [1] and convex graphs. All these classes are factorial. This conclusion can be obtained either by direct counting (such as Cayley’s formula for trees [6]) or can be derived from some more general results. For instance, in [12] it was proved that every subclass of chordal bipartite graphs obtained by forbidding a forest is at most factorial. This result alone implies that chain graphs, bipartite permutation graphs, biconvex and convex graphs are factorial classes (a low bound follows from the fact that each of them contains one of the minimal factorial classes). Also, in [3] it was proved that

Theorem 1. *Every class of bounded clique-width is at most factorial.*

This implies, in particular, that forests and bipartite distance-hereditary graphs are

factorial classes. Indeed, forests are graphs of tree-width 1, and every class of bounded tree-width is also of bounded clique-width (see e.g. [7]). For the class of distance-hereditary graphs an upper bound on the clique-width was shown in [8]. Observe that the class of bipartite distance-hereditary graphs is precisely the class of *domino*-free chordal bipartite graphs [5] (see Figure 2 for the *domino*). Some more classes of chordal bipartite graphs of bounded clique-width can be found in [13].

The above discussion raises the question of whether the chordal bipartite graphs constitute a minimal superfactorial hereditary class. In Section 3, we answer this question negatively by identifying the first proper superfactorial subclass of chordal bipartite. In the attempt to obtain more progress in this direction, in Sections 4 and 5 we study more hereditary subclasses of chordal bipartite graphs. All of them turn out to be factorial. Deriving a lower bound is an easy task, since the list of all minimal factorial classes is finite. For an upper bound, we use Theorem 1 and the following helpful lemma.

Lemma 1. *Let X be a hereditary class. If there is a constant $d \in \mathbb{N}$ and a hereditary class Y with at most factorial speed of growth such that every graph $G = (V, E) \in X$ contains a non-empty subset $A \subseteq V$ such that*

- $G[A] \in Y$,
- each vertex $a \in A$ has either at most d neighbours or at most d non-neighbours in $V - A$,

then X is at most factorial.

Proof. We prove the lemma by induction on $n = |V(G)|$. Let $f(n)$ be the number of n -vertex graphs in X and $f_A(n)$ the number of n -vertex graphs in X with a fixed set A satisfying conditions of the lemma. The value of $f_A(n)$ can be upper bounded as follows:

$$f_A(n) \leq n_1^{cn_1} f(n - n_1) \left(2 \binom{n - n_1}{d} 2^d \right)^{n_1},$$

where $n_1 = |A|$, $n_1^{cn_1}$ is an upper bound on the number of different graphs induced by A (c is a constant associated with the class Y), $f(n - n_1)$ is the number of different graphs induced by $V - A$ and $\left(2 \binom{n - n_1}{d} 2^d \right)^{n_1}$ is an upper bound on the number of ways to place different edges between A and $V - A$. Therefore,

$$f_A(n) \leq n^{cn_1} f(n - n_1) n^{(2d+1)n_1} = n^{tn_1} f(n - n_1), \quad (1)$$

where $t = c + 2d + 1$. By induction $f(n - n_1) < (n - n_1)^{h(n - n_1)}$, for some positive constant h . In order to complete the proof we need to show that $f(n) < n^{hn}$. Without loss of generality, suppose that $h > t + 2$. Taking into account (1) we derive final conclusion:

$$f(n) \leq \sum_{n_1=1}^n \binom{n}{n_1} n^{tn_1} (n - n_1)^{h(n - n_1)} \leq \sum_{n_1=1}^n n^{(t+1)n_1 + h(n - n_1)} < \sum_{n_1=1}^n n^{hn - n_1} < n^{hn}.$$

□

As a special case of this lemma (with $|A| = 1$) we obtain the following corollary.

Corollary 1. *Let X be a hereditary class. If there is a constant d such that every graph $G \in X$ has a vertex of degree at most d or at least $n - d$, then X is at most factorial.*

3 A superfactorial subclass of chordal bipartite graphs

In order to derive a lower bound on the number of n -vertex chordal bipartite graphs Spinrad counted in [16] the number of bipartite adjacency matrices representing these graphs, i.e. binary matrices whose rows correspond to one part of the graph and columns correspond to the other part. In particular, he used in [16] the following construction.

Let M be a $2n$ by $2n$ binary matrix. Divide it into four n by n quadrants. Place an arbitrary perfect matching in the upper left quadrant and a matrix with all values equal 1 in the lower right quadrant. Repeat this construction recursively within the other two quadrants. Let us denote the set of matrices constructed in this way by M^* and the set of bipartite graphs represented by these matrices by \mathcal{Y}^* .

Spinrad showed in [16] that the number of matrices in M^* , and therefore the number of n -vertex graphs in \mathcal{Y}^* , is $\Omega(2^{\Omega(n \log^2 n)})$. He also showed that every graph in \mathcal{Y}^* is chordal bipartite, which implies in particular a superfactorial lower bound for the number of n -vertex chordal bipartite graphs. However, as we show below, not every chordal bipartite graph belongs to $[\mathcal{Y}^*]$.

We denote by $2C_4$ the graph consisting of two disjoint copies of C_4 and by $C_4 - C_4$ the graph obtained from $2C_4$ by adding exactly one edge connecting vertices from different C_4 's.

Lemma 2. *Let G be a graph from \mathcal{Y}^* and C^1 and C^2 two vertex-disjoint induced C_4 in G . Then there are at least two edges between C^1 and C^2 in G .*

Proof. We prove the lemma by induction on the number of vertices in G . Clearly the lemma is true if G contains most 7 vertices.

Let $A \cup B$ be a bipartition of G . By definition, the vertices of G can be partitioned into two parts $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ in such a way that $A_1 \cup B_1$ induces a 1-regular graph and $A_2 \cup B_2$ induces a complete bipartite graph.

The vertices of an arbitrary induced C_4 in G can be arranged within the four subsets of G in exactly one of the following ways:

- (1) one vertex in A_1 , one in B_1 , one in A_2 and one in B_2 ,
- (2) two vertices in A_2 and two in B_2 ,
- (3) one vertex in A_1 , two in B_2 and one in A_2 ,
- (4) one vertex in B_1 , two in A_2 and one in B_2 ,
- (5) two vertices in A_1 and two in B_2 ,
- (6) two vertices in B_1 and two in A_2 .

If both C^1 and C^2 are located according to case 5 (or case 6), then the lemma holds by induction. In all other cases it is easy to check the existence of at least two edges between C^1 and C^2 with endpoints in $A_2 \cup B_2$. \square

Corollary 2. *Every graph in \mathcal{Y}^* is $(2C_4, C_4 - C_4)$ -free.*

Corollary 2 and the lower bound on the number of n -vertex graphs in \mathcal{Y}^* imply the following conclusion.

Theorem 2. *The number of n -vertex $(2C_4, C_4 - C_4)$ -free chordal bipartite graphs is $\Omega(2^{\Omega(n \log^2 n)})$, i.e. the class of $(2C_4, C_4 - C_4)$ -free chordal bipartite graphs is superfactorial.*

4 Excluding a unicyclic graph

As we mentioned in the introduction, excluding from the class of chordal bipartite graphs a forest (i.e. a graph without cycles) results in a factorial class. The results of the previous section show that excluding a graph with two disjoint cycles results in a class which is superfactorial. In the present section, we deal with subclasses of chordal bipartite graphs obtained by excluding a unicyclic graph, i.e. a graph with a single cycle.

The simplest unicyclic chordal bipartite graph is a C_4 . The class of C_4 -free chordal bipartite graphs is precisely the class of forests. By Cayley's formula [6], there are n^{n-2} labeled trees with n vertices, which gives a factorial upper bound on the number of labeled forests. This conclusion can be easily extended to the class of *banner*-free chordal bipartite graphs, where a *banner* is the graph obtained from a C_4 by adding a pendant vertex, i.e. a vertex with exactly one neighbour in the C_4 . Indeed, it is not difficult to see that a connected *banner*-free graph containing a C_4 is complete bipartite. In what follows, we describe two extensions of *banner*-free chordal bipartite graphs and show that each of them is factorial.

4.1 Q -free chordal bipartite graphs

By Q we denote the graph represented in Figure 1 (left) and by $S_{1,2,3}$ the graph represented in Figure 1 (right). The class of $S_{1,2,3}$ -free bipartite graphs was studied in [10], where it was shown to be of bounded clique-width. Therefore, by Theorem 1, it is factorial. Now we use this fact in order to show that the class of Q -free chordal bipartite graphs is factorial.

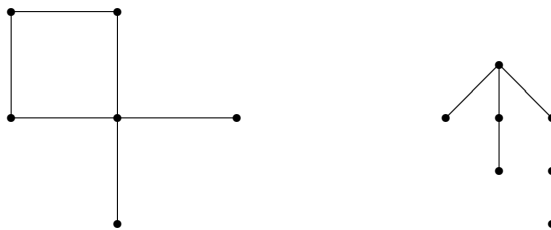


Figure 1: The graphs Q (left) and $S_{1,2,3}$ (right)

Lemma 3. *If a Q -free chordal bipartite graph G contains a C_4 , then it contains a subset U such that $G[U]$ is $S_{1,2,3}$ -free and every vertex of U has at most one neighbour in the rest of the graph.*

Proof. To find a subset U satisfying conditions of the lemma, let us start with a C_4 and extend it to a maximal complete bipartite subgraph H containing it. Denote the parts (color classes) of H by A and B . Also, we denote by C the set of vertices of G outside

H that have neighbours in B , and by D the set of vertices of G outside H that have neighbours in A . We claim that

- (1) *each vertex of C has either exactly one neighbor or exactly one non-neighbour in B .*
 First, we observe that by definition every vertex of C must have both a neighbour and a non-neighbour in B . Now assume that a vertex $c \in C$ has at least two neighbours, say b_1, b_2 , and at least two non-neighbour, say b_3, b_4 , in B . Then for any vertex $a \in A$ the subgraph induced by a, b_1, b_2, b_3, b_4, c is isomorphic to Q .

Similarly, each vertex of D has either exactly one neighbor or exactly one non-neighbour in A . We will say that a vertex of C (D) is of type 1 if it has one neighbor in B (A) and of type 2 if it has one non-neighbour in B (A). If $|A| = |B| = 2$, then every vertex of type 1 in C (D) is also of type 2. To avoid this ambiguity we will assume that every vertex of type 2 has at least two neighbours in B (A). We claim that

- (2) *no vertex of B (A) has both a neighbour of type 1 and a neighbour of type 2 in C (D).* Assume by contradiction that a vertex $b \in B$ has a neighbour c_1 of type 1 and a neighbour c_2 of type 2 in C . Let $b' \in B$ be the non-neighbour of c_2 and a, a' any two vertices in A . Then a, a', b, b', c_1, c_2 induce a Q in G .
- (3) *each vertex of B (A) has at most one neighbour of type 1 or at most one non-neighbour of type 2 in C (D).* Assume a vertex $b \in B$ has two neighbours c_1, c_2 of type 1 in C . Let $b' \in B$ be any vertex of B different from b and a, a' any two vertices in A . Then a, a', b, b', c_1, c_2 induce a Q in G . Now assume b has two non-neighbours c_1, c_2 of type 2 in C . Let $b' \in B$ be any vertex of B different from b and a, a' any two vertices in A . Then a, a', b, b', c_1, c_2 induce a Q in G .

Assume that C has exactly one vertex of type 2. We denote this vertex by c and its only non-neighbour in B by b . If in addition C has a vertex c' of type 1, then by (2) b is the only neighbour of c' in B , and therefore, by (3), C has no other vertices of type 1, i.e. $C = \{c, c'\}$ and every vertex of B has at most one neighbour in C . If C has no vertices of type 2, then also every vertex of B has at most one neighbour in C (by (3)). Similar arguments apply to D . Therefore, if each of C and D contains at most one vertex of type 2, then every vertex of $A \cup B$ has at most one neighbor in the rest of the graph, in which case the lemma is true.

Now suppose that one of C, D has at least two vertices of type 2. Without loss of generality let it be C . Then

- (4) *C has no vertices of type 1.* Let c_1, c_2 be two vertices of type 2 in C , $b_1 \in B$ the non-neighbour of c_1 and $b_2 \in B$ the non-neighbour of c_2 . Assume by contradiction that C contains a vertex c_3 of type 1 with the only neighbour b_3 in B . By (2) b_3 must be equal both to b_1 and b_2 , but then $b_1 = b_2$ contradicts (3).
- (5) *every vertex of C is adjacent to every vertex of D .* To show this, assume a vertex $c \in C$ is not adjacent to a vertex $d \in D$. By (4) c is of type 2. We denote any two neighbors of c in B by b_1, b_2 and its only non-neighbour in B by b_3 . Also, let a be any neighbour of d in D . The vertices a, b_1, b_2, b_3, c, d induce a Q in G .

- (6) *every vertex of $U = A \cup B \cup C \cup D$ has at most one neighbour in $V - U$.* Indeed, by definition the vertices of $A \cup B$ have no neighbours in $V - U$. Suppose now that a vertex $c \in C$ has two neighbours $x, y \in V - U$. Then c, x, y together with any two neighbours of c in B and any vertex of A induce a Q . Finally, let d be any vertex of D with two neighbours $x, y \in V - U$. By assumption, C is not empty, say $c \in C$, and by (5) d is adjacent to every vertex of C . Also we know that d must have at least one neighbour a in A , and c has a neighbour b in B . Then a, b, c, d, x, y induce a Q .
- (7) *if D has a vertex d of type 2, then D has no vertices of type 1.* By contradiction, let d' be a vertex of type 1 in D . We denote by a, a' any two neighbours of d in A , and by c, c' any two vertices of C . By (2) d' is adjacent neither to a nor to a' , and by (5) c, c', d, d' induced a C_4 . But then a, a', c, c', d, d' induce a Q .

From the above discussion it follows that

- either all vertices of C and D are of type 2
- or all vertices of C are of type 2 and all vertices of D are of type 1.

In both cases, $U = A \cup B \cup C \cup D$ induces an $S_{1,2,3}$ -free bipartite graph (to better see this, consider the bipartite complement of $G[U]$). Together with (6) this completes the proof. \square

Theorem 3. *The class of Q -free chordal bipartite graphs is factorial.*

Proof. Since the class of Q -free chordal bipartite graphs is an extension of forests, it is at least factorial. For an upper bound, we apply Lemma 1 with Y being the union of the class of forests and the class of $S_{1,2,3}$ -free bipartite graphs. Since both classes are factorial, Y is factorial too. As to the set A satisfying conditions of Lemma 1, it is either $V(G)$ if G is C_4 -free or a set U defined in Lemma 3. \square

4.2 A -free chordal bipartite graphs

In this section, we show that the class of A -free chordal bipartite graphs is factorial, where A is the graph represented in Figure 2. Again, this class contains all forests, and therefore, it is at least factorial. To derive a factorial upper bound, we will prove a stronger result: we will show that the clique-width of graphs in this class is bounded by a constant. To this end, we need to fix some terminology.

Given a graph G , a subset $U \subset V(G)$ and a vertex $x \notin U$, we say that x *distinguishes* U if it has both a neighbour and a non-neighbour in U . A subset of vertices of G indistinguishable by the vertices outside the subset is called a *module* of G . A module is *trivial* if it consists of a single vertex of G or includes all its vertices. Finally, G is said to be *prime* if each of its modules is trivial. In particular, every prime graph with at least 3 vertices is connected. In the class of A -free chordal bipartite graphs the structure of prime graphs can be described as follows.

Lemma 4. *Every prime A -free chordal bipartite graph with at least three vertices is either a tree or a the domino.*

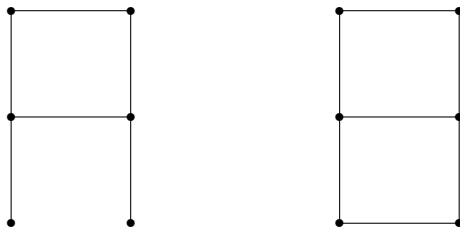


Figure 2: The graphs A (left) and $domino$ (right)

Proof. Let G be a prime A -free chordal bipartite graph with at least three vertices. If G is C_4 -free, then it is a tree, since it is a connected graph without cycles. Therefore, assume G contains a C_4 . First, we extend this C_4 to a maximal complete bipartite subgraph H containing it. Let A and B be the two parts of H . Notice that $|A| \geq 2$ and $|B| \geq 2$, since H contains a C_4 . Now we denote by C the set of vertices of G outside H that have neighbours in B , and by D the set of vertices of G outside H that have neighbours in A . Then we claim that

- (1) $C \neq \emptyset$ and $D \neq \emptyset$. Indeed, if C is empty, then no vertex of G distinguishes B , in which case B is a non-trivial module of G , contradicting primality of G . Similarly, D is not empty.
- (2) $C \cup D$ induces a complete bipartite graph in G . Indeed, assume by contradiction that a vertex $c \in C$ is not adjacent to a vertex $d \in D$. By definition of C , vertex c must have a neighbour b_1 in B , and by definition of H , it must have a non-neighbour b_0 in B (since otherwise H is not maximal). Similarly, d must have a neighbour a_1 and a non-neighbour a_0 in A . But then vertices a_1, a_0, b_1, b_0, c, d induce an A in G .
- (3) $V(G) = A \cup B \cup C \cup D$. Indeed, if the vertex set of G contains more vertices, then there must exist a vertex x that has a neighbour in $C \cup D$. Assume x is adjacent to a vertex $d \in D$. Let a be a neighbour of d in A . Also, let c be any vertex in C , $b_1 \in B$ a neighbour and $b_0 \in B$ a non-neighbour of c . Now a, b_1, b_0, c, d, x induce an A in G .

Now we look at the subgraph of G induced by A and D . Let x and y be two vertices of D . We denote by X the set of neighbours of x in A and by Y the set of neighbours of y in A . Observe that by definition of D both sets X and Y are non-empty. Then

- (4) $X \neq Y$, since otherwise $\{x, y\}$ is a non-trivial module of G .
- (5) if the intersection $X \cap Y$ is not empty, then $X \cup Y = A$. Assume $X \cap Y$ contains a vertex a_1 , and suppose by contradiction that there is a vertex $a_0 \in A$ non-adjacent both to x and y . Since $X \neq Y$, there must exist a vertex $a_2 \in A$ adjacent to one of x, y and non-adjacent to the other. But now the vertices a_0, a_1, a_2, x, y together with any vertex of B induce an A in G .
- (6) neither $X \subset Y$ nor $Y \subset X$. Indeed, if, say, $X \subset Y$, then by the previous claim Y must coincide with A , but this is a contradiction to the maximality of H .

- (7) if X intersects Y , then X intersects the neighbourhood of any other vertex of D in A . Assume by contradiction that X intersects Y , but X is disjoint from Z , where Z is the neighbourhood of a vertex $z \in D$ in A . Then from (5) it follows that $Z \subset Y$, contradicting (6).

From (7) it follows that the neighbourhoods of vertices of D in the set A are either pairwise intersecting or pairwise disjoint. Moreover, if in the graph induced by $A \cup D$ the neighbourhoods are pairwise intersecting, then according to (5) in the bipartite complement to this graph the neighbourhoods are pairwise disjoint. Therefore, to understand the structure of this graph, it is sufficient to analyze the case of pairwise disjoint neighbourhoods. Under this assumption the neighbourhood of each vertex of D in the set A creates a module and therefore it must consist of a single vertex.

Assume D contains two vertices, say x and y . We denote by a_1 the only neighbour of x in A and by a_2 the only neighbour of y in A . Also, let c be an arbitrary vertex of C and b is an arbitrary non-neighbour of c in B . Then a_1, a_2, b, c, x, y induce a C_6 , which is not possible, since G is a chordal bipartite graph. Moreover, the same arguments work if we consider the bipartite complement of the graph induced by $A \cup D$, since the vertices a_1, a_2, x, y induce a $2K_2$ and this graph is self-complementary in the bipartite sense. Therefore, regardless of whether the neighbourhoods of vertices of D in the set A are pairwise intersecting or pairwise disjoint, we conclude that $|D| = 1$. Similarly, $|C| = 1$. This implies that $|A| = |B| = 2$, since otherwise G is not prime. But then the set $A \cup B \cup C \cup D$ induces a domino. \square

Theorem 4. *The clique-width of A -free chordal bipartite graphs is at most 6.*

Proof. It is known (see e.g. [7]) that the clique-width of a graph G equals the maximum of clique-width taken over all prime induced subgraphs of G . Also, the clique-width of any tree is at most 3 (again see e.g. [7]) and the clique-width of a domino is at most 6, since the clique-width cannot exceed the number of vertices. Hence the theorem. \square

Combining the above theorem with Theorem 1 we arrive at the final conclusion.

Corollary 3. *The class of A -free chordal bipartite graphs is factorial.*

5 $K_{p,p}$ -free and more general chordal bipartite graphs

In this section, we show that for any positive constant $p \geq 2$ the class of $(K_{p,p} + K_1)$ -free chordal bipartite graphs is factorial. We start with the base case of $K_{p,p}$ -free chordal bipartite graphs and show a stronger result for them, namely, we prove that the tree-width of $K_{p,p}$ -free chordal bipartite graphs is bounded by a constant. To this end, let us first introduce some terminology and auxiliary results related to the notion of tree-width.

Definition 1. *A chordal graph (or triangulated graph) is a graph with no chordless cycle of length ≥ 4 .*

Definition 2. *A triangulation of a graph G is any chordal graph H containing G as a spanning subgraph, i.e. $V(H) = V(G)$ and $E(H) \supseteq E(G)$.*

Clearly every graph has a triangulation, since every graph is a spanning subgraph of a complete graph on the same vertex set. When determining the tree-width of a graph, we are interested in triangulations with the smallest possible size of a maximum clique, which is due to the following well-known lemma (see e.g. [9]).

Lemma 5. *The tree-width of G is at most k if and only if there is a triangulation of G with maximum clique size at most $k + 1$.*

Let $G = (X, Y, E)$ be a bipartite graph. We call the sets X and Y the color classes of G . An arbitrary complete bipartite subgraph of G will be denoted $M = (A, B)$, i.e. M is the graph with vertex set $A \cup B$ and edge set $E = \{(a, b) \mid a \in A, b \in B\}$. We will assume, by definition, that both A and B have size at least 2. If (A, B) is a complete bipartite subgraph of G and H is a triangulation of G , then either $H[A]$ or $H[B]$ is a complete subgraph of H .

Now let G be a chordal bipartite graph. Denote by \mathcal{M} the set of all maximal complete bipartite subgraphs (A, B) of G (with $|A| \geq 2$ and $|B| \geq 2$) and let \mathcal{C} be a set containing one of the color classes for each graph $(A, B) \in \mathcal{M}$.

We say that a graph $M_1 = (A_1, B_1)$ crosses a graph $M_2 = (A_2, B_2)$ from left (from right) if $A_2 \subseteq A_1$ and $B_1 \subseteq B_2$ ($A_1 \subseteq A_2$ and $B_2 \subseteq B_1$). We also say that a set \mathcal{C} is *feasible* if for each pair $M_1 = (A_1, B_1), M_2 = (A_2, B_2) \in \mathcal{M}$ such that $M_1 = (A_1, B_1)$ crosses a graph $M_2 = (A_2, B_2)$ from left, either A_1 or B_2 is not in \mathcal{C} .

For a feasible set \mathcal{C} , let $H_{\mathcal{C}}$ denote the graph obtained from G by completing each $C \in \mathcal{C}$, i.e. by adding all possible edges connecting vertices of C . The following results have been proved in [9].

Theorem 5. *If \mathcal{C} is a feasible set of color classes of a chordal bipartite graph G , then $H_{\mathcal{C}}$ is a chordal graph, i.e. a triangulation of G .*

Theorem 6. *Let K be a maximal clique in $H_{\mathcal{C}}$ with $|K| > 2$. Let $K_x = K \cap X$ and $K_y = K \cap Y$. Assume $|K_x| \geq 2$. Then one of the following two cases holds:*

1. $|K_y| = 1$ and there exists a maximal complete bipartite subgraph (A, B) such that $K_x = A, y \in B$ and $A \in \mathcal{C}$.
2. $|K_y| > 1$ and there exist maximal complete bipartite subgraphs (A_1, B_1) and (A_2, B_2) , with $A_1 \in \mathcal{C}$ and $B_2 \in \mathcal{C}$ such that $K_x \subseteq A_1$ and $K_y \subseteq B_2$.

5.1 $K_{p,p}$ -free chordal bipartite graphs

Throughout this section $G = (X, Y, E)$ is a $K_{p,p}$ -free chordal bipartite graph. Therefore, for any complete bipartite subgraph (A, B) of G we have either $|A| < p$ or $|B| < p$. Our goal is to prove that the tree-width of G is at most $2p - 3$. We start with the following lemma.

Lemma 6. *For every $K_{p,p}$ -free chordal bipartite graph G there exists a feasible set \mathcal{C} such that $|C| < p$ for each $C \in \mathcal{C}$.*

Proof. Given the collection $\mathcal{M} = \{M_1, \dots, M_m\}$ of all maximal complete bipartite subgraphs of G , we construct the set $\mathcal{C} = \{C(M_1), \dots, C(M_m)\}$ step by step starting with

$\mathcal{C}_0 = \emptyset$. In the k -th step we add to \mathcal{C}_{k-1} a set $C(M_k) \in \{A_k, B_k\}$ such that $|C(M_k)| < p$ and $\mathcal{C}_k = \mathcal{C}_{k-1} \cup \{C(M_k)\}$ satisfies the condition of feasibility. In the first step, we take the smallest of A_1 and B_1 . Since G is $K_{p,p}$ -free, we have $|C(M_1)| < p$, and since \mathcal{C}_1 consists of a single set, it is obviously feasible.

Suppose we have successfully made $k-1$ steps. If M_k does not cross any of M_1, \dots, M_{k-1} , then we simply include in \mathcal{C}_k the smallest of A_k and B_k . Now suppose that M_k crosses $s \geq 0$ maximal complete bipartite graphs M_{l_1}, \dots, M_{l_s} from left and $t \geq 0$ maximal complete bipartite graphs M_{r_1}, \dots, M_{r_t} from right, with $s + t \leq k - 1$. That is, for any $i = 1, \dots, s$ and any $j = 1, \dots, t$:

$$\begin{aligned} A_{l_i} &\subseteq A_k \subseteq A_{r_j} \\ B_{r_j} &\subseteq B_k \subseteq B_{l_i} \end{aligned} \tag{2}$$

From (2) it follows that M_{r_j} cross M_{l_i} from left, which means that A_{r_j} and B_{l_i} cannot both belong to \mathcal{C}_{k-1} , since otherwise \mathcal{C}_{k-1} does not satisfy the condition of feasibility. This leaves us with three possible situations:

1. There exists j such that $A_{r_j} \in \mathcal{C}_{k-1}$ and for all $i = 1, \dots, s$, $B_{l_i} \notin \mathcal{C}_{k-1}$. In this case we define $C(M_k) = A_k$. Since none of B_{l_i} is in \mathcal{C}_k , the feasibility condition is not violated, and since $A_k \subseteq A_{r_j}$ and $A_{r_j} \in \mathcal{C}_{k-1}$, we have $|A_k| \leq |A_{r_j}| < p$.
2. There exists i such that $B_{l_i} \in \mathcal{C}_{k-1}$ and for all $j = 1, \dots, t$, $A_{r_j} \notin \mathcal{C}_{k-1}$. In this case we define $C(M_k) = B_k$. Since none of A_{r_j} is in \mathcal{C}_k , the feasibility condition is not violated, and since $B_k \subseteq B_{l_i}$ and $B_{l_i} \in \mathcal{C}_{k-1}$, we have $|B_k| \leq |B_{l_i}| < p$.
3. For each $i = 1, \dots, s$ and each $j = 1, \dots, t$, both $A_{l_i} \in \mathcal{C}_{k-1}$ and $B_{r_j} \in \mathcal{C}_{k-1}$. In this case we define $C(M_k)$ to be the smallest of A_k and B_k . Since none of A_{r_j} is in \mathcal{C}_k and none of B_{l_i} is in \mathcal{C}_k , the feasibility condition is not violated, and since G is $K_{p,p}$ -free, we have $|C(M_k)| < p$.

By induction we conclude that the above procedure constructs a feasible set \mathcal{C} such that for every $C(M) \in \mathcal{C}$, $|C(M)| < p$. \square

Lemma 7. *The tree-width of $K_{p,p}$ -free chordal bipartite graph is at most $2p - 3$.*

Proof. Let G be a $K_{p,p}$ -free chordal bipartite graph and \mathcal{C} be a feasible set for this graph constructed according to Lemma 6, i.e. for each $C \in \mathcal{C}$, $|C| < p$. By Theorem 5, G is a subgraph of the chordal graph $H_{\mathcal{C}}$ obtained from G by making each $C \in \mathcal{C}$ complete. From Theorem 6 it follows that the size of a maximum clique in $H_{\mathcal{C}}$ is at most $2p - 2$. In conjunction with Lemma 5 it means that the tree-width of G is at most $2p - 3$. \square

Lemma 7 leads to a number of important conclusions. First of all, together with an upper factorial bound on the number of graphs of bounded clique- (and therefore, tree-) width and a lower factorial bound on the number of $K_{2,2}$ -free chordal bipartite graphs (forests), Lemma 7 implies the following result.

Theorem 7. *For every integer $p \geq 2$, the class of $K_{p,p}$ -free chordal bipartite graphs is factorial.*

Also, since the tree-width of $K_{n,n}$ is n , Lemma 7 provides a complete characterization of hereditary classes of chordal bipartite graphs of bounded tree-width.

Theorem 8. *A hereditary subclass X of chordal bipartite graphs is of bounded tree-width if and only if the set of forbidden induced subgraphs for X contains a $K_{p,q}$ for some positive integers p, q .*

5.2 $(K_{p,p} + K_1)$ -free chordal bipartite graphs

In this section, we extend the result of Theorem 7 from $K_{p,p}$ to $(K_{p,p} + K_1)$ -free chordal bipartite graphs. We need two auxiliary results. The first of them is an easy adaption of Corollary 1 to the case of bipartite graphs.

Lemma 8. *Let B be a hereditary class of bipartite graphs. If there is a constant d such that every graph $G \in B$ contains a vertex which has either at most d neighbours or at most d non-neighbours in the opposite part of the graph, then B is at most factorial.*

The proof of one more auxiliary result can be found in [18]. To make the paper self-contained we present it here.

Lemma 9. *Let U be a set with $|U| = n$ and let A_1, \dots, A_q be subsets of U such that $|A_1| = \dots = |A_q| \geq \left\lceil \frac{(2^{q-1}-1)n+t}{2^{q-1}} \right\rceil$, $n > t, q \geq 2$. Then $|\bigcap_{i=1}^q A_i| \geq t$.*

Proof. We prove the lemma by induction on q . For $q = 2$, we have $|A_1| = |A_2| \geq \left\lceil \frac{n+t}{2} \right\rceil$, $|A_1 \cup A_2| \leq n$. Therefore, $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2| \geq 2 \left\lceil \frac{n+t}{2} \right\rceil - n \geq n+t-n = t$.

Now assume that the lemma is valid for any $q-1$ subsets A_1, \dots, A_{q-1} of U . Denote $B = A_1 \cap \dots \cap A_{q-1}$. Since $|A_i| \geq \left\lceil \frac{(2^{q-1}-1)n+t}{2^{q-1}} \right\rceil = \left\lceil \frac{(2^{q-2}-1)n + \frac{n+t}{2}}{2^{q-2}} \right\rceil$, we have by induction that $|B| \geq \frac{n+t}{2}$. Moreover, since $|B|$ is an integer number, we have $|B| \geq \left\lceil \frac{n+t}{2} \right\rceil$. Now from the inequality $|A_q| \geq \left\lceil \frac{(2^{q-1}-1)n+t}{2^{q-1}} \right\rceil \geq \left\lceil \frac{n+t}{2} \right\rceil$ we derive that $|B \cap A_q| = |\bigcap_{i=1}^q A_i| \geq t$. \square

Theorem 9. *For any fixed integer $p \geq 2$, the class of $(K_{p,p} + K_1)$ -free chordal bipartite graphs is factorial.*

Proof. For any $p \geq 2$, the class of $(K_{p,p} + O_1)$ -free chordal bipartite graphs contains the class of forests, which proves the lower bound.

For an upper bound, let $s = p(2^{p-1} + 1)$ and assume that a $(K_{p,p} + O_1)$ -free chordal bipartite graph $G = (U, V, E)$ contains $K_{s,s}$ as an induced subgraph. Partition $U = A \cup C$ and $V = B \cup D$ in such a way that $A \cup B$ induces a $K_{s,s}$.

- (1) *Each vertex of C (of D) has at most $p-1$ non-neighbours in B (in A).* If $x \in C$ has p non-neighbours $b_1, \dots, b_p \in B$, then x together with $\{b_1, \dots, b_p\}$ and any p vertices from A induce a $K_{p,p} + K_1$, which is a contradiction. The second case is analogous.
- (2) *Each vertex of C (of D) has at most $p-1$ non-neighbours in D (in C).* Assume that $x \in C$ has p non-neighbours in D which we denote by d_1, \dots, d_p . By (1) for each $i = 1, \dots, p$, $|N(d_i) \cap A| \geq s - p + 1 \geq \left\lceil \frac{(2^{p-1}-1)s+p}{2^{p-1}} \right\rceil$. This inequality together with Lemma 9 imply that $|\bigcap_{i=1}^p (N(d_i) \cap A)| \geq p$. In other words, there exist p vertices $\{a_1, \dots, a_p\} \subset A$ such that the set $\{x, a_1, \dots, a_p, d_1, \dots, d_p\}$ induces a $K_{p,p} + O_1$. This contradiction shows that x has at most $p-1$ non-neighbours in D .

For a positive integer t , let us denote by X_t the class of $K_{t,t}$ -free chordal bipartite graphs and by B_t the set of all bipartite graphs containing a vertex with at most t neighbours or at most t non-neighbours in the opposite part. Then the class of $(K_{p,p} + K_1)$ -free chordal bipartite graphs is a subclass of $X_s \cup [B_{2p-2}]$. Indeed, if a $(K_{p,p} + K_1)$ -free chordal bipartite graph G is not in X_s , then it contains a $K_{s,s}$, in which case G belongs to B_{2p-2} by (1) and (2) (if the set $C \cup D$ is non-empty, then any of its vertex has at most $2p - 2$ non-neighbours in the opposite part; if $C \cup D$ is empty, then each vertex of G has $0 \leq 2p - 2$ non-neighbours in the opposite part). Obviously, the same is true for every induced subgraph of G , since deletion of a vertex from G cannot increase the number of non-neighbours of the remaining vertices. Therefore, G belongs to $[B_{2p-2}]$. The class X_s is at most factorial by Theorem 7 and the class $[B_{2p-2}]$ is at most factorial by Lemma 8. Therefore, the class of $(K_{p,p} + K_1)$ -free chordal bipartite graphs is at most factorial as well. \square

6 Open problems

In this paper, we proved that the class of chordal bipartite graphs is not a minimal superfactorial class and revealed a number of new factorial members in the family of hereditary subclasses of chordal bipartite graphs. However, the most important question of finding a minimal superfactorial class in this family remains open. At present, the only candidate for this role is the class Y^* (more precisely, its hereditary closure $[Y^*]$) described recursively in Section 3. Whether it is a minimal superfactorial class is a challenging research problem. Also, it would be interesting to characterize this class in terms of minimal forbidden induced subgraphs. Finally, identifying more (candidates for) minimal superfactorial classes is a question of great importance and, apparently, of great difficulty, since none has been identified so far.

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